Topology

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) A topological space

- ○ is defined by a set, a topology, and an atlas.
- ○ is a set without any further structure.
- ○ defines a notion of open sets.
- ○ always has integer dimension.
- ○ allows to check the continuity of a map from the underlying set to itself.

b) The chaotic topology on a set M

- ○ cannot be defined on the natural numbers $\mathbb{N}$.
- ○ consists of all subsets of M.
- ○ contains the empty set.
- ○ is the coarsest topology on M.
- ○ makes all maps $f : N \to M$ continuous, where the domain may carry an arbitrary topology.

c) Consider a map $f : M \to N$ between topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$.

- ○ Continuity can only be defined if $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ for positive integers m and n.
- ○ For some maps, one can arrange for the topological notion of continuity to coincide with the undergraduate analysis notion of continuity.
- ○ Continuity of a map can only be defined for some topologies.
- ○ Continuity is a property of a map that only depends on the topology $\mathcal{O}_M$.
- ○ Choosing the discrete topology on M makes all maps from M to N continuous.

d) A subset $U \subseteq M$ of a topological space $(M, \mathcal{O})$

- ○ may be open and not open at the same time.
- ○ may be open, but not closed.
- ○ may be closed, but not open.
- ○ may be open and closed.
- ○ may be not open and not closed.

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Exercise 2: Topologies on a simple set

This exercise is about recognizing topologies.

Question: Write down the definition of a topology $\mathcal{O}$ on a set M.

Solution:

i.

ii.

iii.

Let $M = \{1, 2, 3, 4\}$ be a set.

Question: Does $\mathcal{O}_1 := \{\emptyset, \{1\}, \{1, 2, 3, 4\}\}$ constitute a topology on M?

Solution:

Question: What about $\mathcal{O}_2 := \{\emptyset, \{1\}, \{2\}, \{1, 2, 3, 4\}\}$?

Solution:

Question: Are there other topologies than the ones recognized so far?

Exercise 3: Continuous functions

Becoming familiar with the striking impact of choosing topologies on the continuity of a map.

Question: What is the definition of a continuous map?

Solution:

Question: Let $M = \{1, 2, 3, 4\}$ and consider the identity map $\mathrm{id}_M : M \to M$ defined by

$$\mathrm{id}_M(1) = 1\,, \quad \mathrm{id}_M(2) = 2\,, \quad \mathrm{id}_M(3) = 3\,, \quad \mathrm{id}_M(4) = 4\,.$$

Is the map id_M continuous if the domain is equipped with the chaotic topology and the target with the topology $\mathcal{O}_{\text{target}} := \{\emptyset, \{1\}, \{1, 2, 3, 4\}\}$?

Solution:

Question: Consider the inverse $\mathrm{id}_M^{-1} : M \to M$ of the identity map id_M, such that now the target is equipped with the chaotic topology and the domain with the topology $\{\emptyset, \{1\}, \{1, 2, 3, 4\}\}$.

Provide the values of the map id_M^{-1} and decide whether id_M^{-1} is continuous!

Solution:

$$\mathrm{id}_M^{-1}(1) = \qquad \mathrm{id}_M^{-1}(2) = \qquad \mathrm{id}_M^{-1}(3) = \qquad \mathrm{id}_M^{-1}(4) =$$

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Exercise 4: The standard topology on $\mathbb{R}^d$

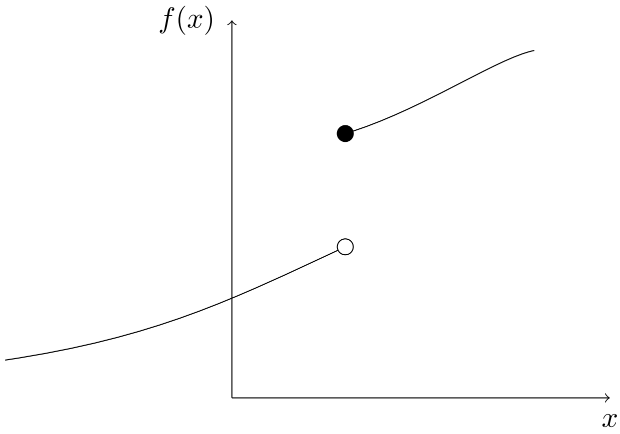
The topology you always knew, possibly without knowing.

Question: Sketch the real intervals below and decide whether they are open or not in $\mathcal{O}_{\text{standard}}$!

interval	**sketch**	**open or not open in $\mathcal{O}_{\text{standard}}$?**
$(0,1)$		
$[0,1)$		
$(0,1]$		
$[0,1]$		
$(0,1) \cup (2,3)$		

Question: Which of the following subsets of $\mathbb{R}^2$ are open with respect to the standard topology?

Question: Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by the following graph.



If domain and target of the map are both equipped with the standard topology, is this function continuous?

Solution: