Topological manifolds

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) Topological manifolds

- ○ have an integer dimension.
- ○ are equipped with a topology and exactly one chart.
- ○ are a special case of topological spaces.
- ○ are the minimal structure required to define continuity of maps.
- ○ are homeomorphic to $\mathbb{R}^d$.

b) Which statements about topological manifolds are correct?

- ○ Any topological manifold has a maximal atlas.
- ○ A maximal atlas on a topological manifold may contain infinitely many charts.
- ○ Continuity of a curve on a topological manifold can be checked with respect to an atlas.
- ○ An atlas of a topological manifold can never contain only one chart.
- ○ If a function is continuous in one chart, it is continuous in every chart of a maximal atlas of a topological manifold.

c) It is correct that

- ○ the real line $(\mathbb{R}, \mathcal{O}_s, \mathcal{A}_\mathbb{R})$ equipped with the standard topology and an atlas whose only chart is the identity map over $(\mathbb{R}, \mathcal{O}_s)$ is a topological manifold of dimension 1.
- ○ a topological manifold can have a finite set underlying.
- ○ a topological manifold can never have a topology that is a subset topology.
- ○ for the topological space $(S^{42}, \mathcal{O}_s|_{S^{42}})$, you cannot build an atlas.
- ○ a function $f : M \to \mathbb{R}$ can only be said to be continuous if M is a topological manifold.

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Exercise 2: An Atlas from a Real World – the Moebius river

How to chart the Moebius strip

Question: Consider the Moebius strip you received as the first "Bastelset" for this tutorial that has a river printed on it. How many charts do you need to cover the Moebius strip?

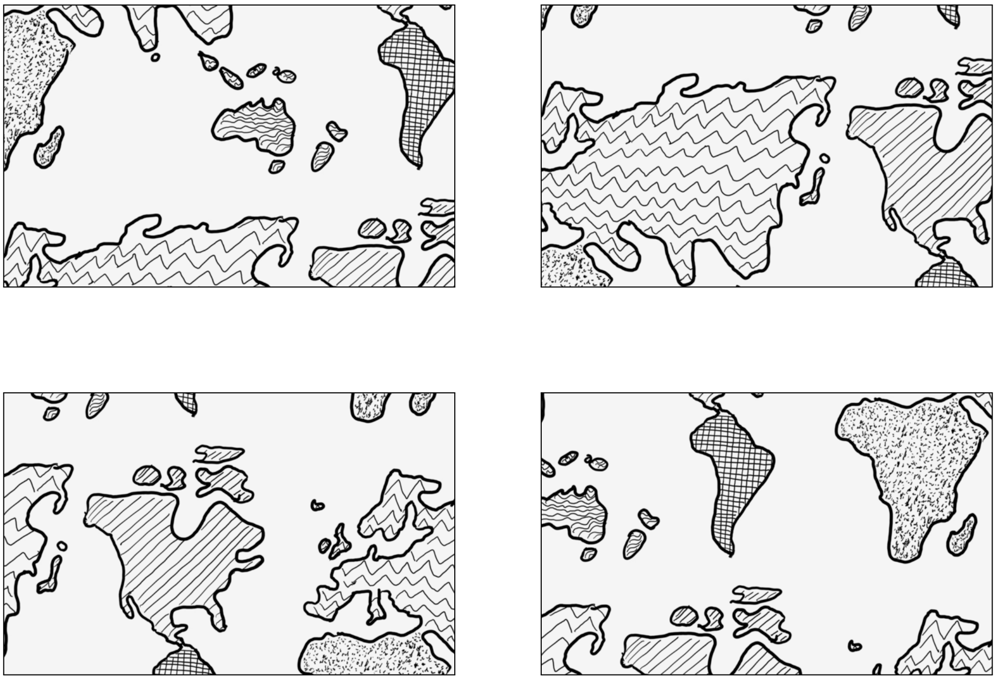
Draw the image of the river on the Moebius strip under the chart map(s)!

Solution:

Exercise 3: A Real World from an Atlas

Getting familiar with reconstructing manifolds from its charts.

Question: You received an atlas containing four charts—shown below—of some Real World. Reconstruct the manifold described by this atlas by mentally gluing together the pieces of the second “Bastelset” in the appropriate overlap regions!



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Exercise 4: Before the invention of the wheel

Another one-dimensional topological manifold. Another one?

Consider the set $F^1 := \{(m,n) \in \mathbb{R}^2 \mid m^4 + n^4 = 1\}$ of pairs of real numbers (m,n). It be equipped with the subset topology $\mathcal{O}_s|_{F^1}$ inherited from the standard topology on $\mathbb{R}^2$.

Question: We look at a map $x : F^1 \to \mathbb{R}$ that maps a pair in F^1 to the first entry of the pair. Write this in formal mathematical terms! Is this map injective?

Solution:

Question: This map may be made injective by restricting its domain to either of two maximal open subsets of F^1. Which ones?

Solution:

Question: Now, construct an injective map $y : F^1 \to \mathbb{R}$ that maps every pair in a maximal open subset of F^1 to the *second* entry of the pair.

Solution:

Question: Is this map y invertible? If so, construct the inverse y^{-1}!

Solution:

Question: Do the domains of the maps x and y overlap? Do the map x and y have overlap in their domain? If so, construct the *transition map* $x \circ y^{-1}$ and specify its domain and target!

Solution:

Question: How many maps (constructed this way) do you need for their domains to cover the whole set F^1? Does the collection of these domains and maps form an atlas of F^1?

Solution: