THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

GRAVITY AND LIGHT

Multilinear Algebra

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) What statements on vector spaces are correct?

- ○ Commutativity of multiplication is a vector space axiom.
- ○ Every vector is a matrix with only one column.
- ○ Every linear map between vector spaces can be represented by a unique quadratic matrix.
- ○ Every vector space has a corresponding dual vector space.
- ○ The set of everywhere positive functions on $\mathbb{R}$ with pointwise addition and S-multiplication is vector space.

b) What is true about tensors and their components?

- ○ The tensor product of two tensors is a tensor.
- ○ You can always reconstruct a tensor from its components and the corresponding basis.
- ○ The number of indices of the tensor components depends on dimension.
- ○ The Einstein summation convention does not apply to tensor components.
- ○ A change of basis does not change the tensor components.

c) Given a basis for a d-dimensional vector space V,

- ○ one can find exactly d^2-different dual bases for the corresponding dual vector space V^*.
- ○ by removing one basis vector of the basis of V, a basis for a $(d-1)$-dimensional vector space V_1 is obtained.
- ○ the continuity of a map $f \colon V \to W$ depends on the choice of basis for the vector space W.
- ○ one can extract the components of the elements of the dual vector space V^*.
- ○ each vector of V can be reconstructed from its components.

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Exercise 2: Vector spaces

Building the vector space $\mathbb{R}^3$ and its dual.

Let $V := \mathbb{R}^3$ be a set of all real triples.

Question: We equip the set V with addition $\oplus : V \times V \longrightarrow V$ and S-multiplication $\odot : \mathbb{R} \times V \longrightarrow V$, defined by

$$(a, b, c) \oplus (d, e, f) := (a + d, b + e, c + f)$$

and

$$\lambda \odot (a, b, c) := (\lambda \cdot a, \lambda \cdot b, \lambda \cdot c),$$

where $+$ and $\cdot$ are the addition and multiplication on $\mathbb{R}$. Check that $(V, \oplus, \odot)$ is a vector space.

Solution:

Question: Consider the map

$$d : V \longrightarrow V;\ (a, b, c) \mapsto d((a, b, c)) := (b, 2c, 0).$$

Is d linear?

Solution:

Question: Show that $d \circ d$ is linear.

Solution:

Question: Consider the map

$$i : V \longrightarrow \mathbb{R};\ (a, b, c) \mapsto i((a, b, c)) := a + \frac{1}{2}b + \frac{1}{3}c.$$

Check linearity. Of what set is i an element?

Solution:

Question: Consider the map

$$G : V \times V \longrightarrow \mathbb{R}$$

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto 2 \cdot a_1 \cdot a_2 + \frac{2}{3} \cdot a_1 \cdot c_2 + \frac{2}{3} \cdot b_1 \cdot b_2 + \frac{2}{3} \cdot c_1 \cdot a_2 + \frac{2}{5} c_1 \cdot c_2.$$

Show that G is multilinear.

Solution:

Question: Compare the above map $d : V \longrightarrow V$ with the map $\delta : P \longrightarrow P$ from the lecture and construct a bijective linear map $j : P \longrightarrow \mathbb{R}^3$ such that

$$d = j \circ \delta \circ j^{-1}.$$

Solution:

Exercise 3: Indices

Winning the index battle.

Let V be a d- dimensional vectorspace. Consider two tensors A and B, where

$$A : V^* \times V^* \xrightarrow{\sim} \mathbb{R}$$

and

$$B : V \times V \xrightarrow{\sim} \mathbb{R} \ .$$

V has the basis $e_1, e_2, \ldots, e_d$ and V^* has the basis $\epsilon^1, \epsilon^2, \ldots, \epsilon^d$.

Question: Define the components A^{ab} of A and B_{ab} of B with respect to the given bases.

Solution:

Question: Recall the "Einstein summation convention" and write it down in your own words.

Solution:

Question: We define $A^{[ab]} := \frac{1}{2}\left(A^{ab} - A^{ba}\right)$. Show that

$$A^{[ab]} = -A^{[ba]}$$

and also

$$A^{[ab]}B_{ab} = A^{ab}B_{[ab]}.$$

Solution:

Question: We additionally define $B_{(ab)} := \frac{1}{2}(B_{ab} + B_{ba})$. Now, show that

$$B_{(ab)} = B_{(ba)}$$

and again

$$A^{ab} B_{(ab)} = A^{(ab)} B_{ab}.$$

Solution:

Question: Using the results from the previous questions, we can easily show

$$A^{[ab]} B_{(ab)} = 0,$$

i.e., the summation (contraction) of symmetric and anti-symmetric indices yields zero.

Solution:

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Exercise 4: Linear maps as tensors

Recognizing that a linear map $V^* \xrightarrow{\sim} V^*$ is a $(1,1)$-tensor over V.

Question: Given a vector space V and linear map $\phi : V^* \xrightarrow{\sim} V^*$ construct a $(1,1)$-tensor T_ϕ.

Solution:

Question: Given a $(1,1)$-tensor $T : V^* \times V \xrightarrow{\sim} \mathbb{R}$, construct a linear map $\phi_T : V^* \xrightarrow{\sim} V^*$.

Solution:

Question: Show that

a) $T_{\phi_T} = T$

b) $\phi_{T_\phi} = \phi$

Solution:

Question: Conclude that a linear map $\phi : V^* \xrightarrow{\sim} V^*$ is a $(1,1)$-tensor.

Solution: