Differentiable Manifolds

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) The function $f : \mathbb{R} \to \mathbb{R} \ldots$

- ○ …, defined by $f(x) = x^2$, lies in $C^3(\mathbb{R} \to \mathbb{R})$.
- ○ …, defined by $f(x) = |x|$, lies in $C^2(\mathbb{R} \to \mathbb{R})$.
- ○ …, defined by $f(x) = |x^3|$, lies in $C^3(\mathbb{R} \to \mathbb{R})$.
- ○ …, defined by $f(x) = \exp|x|$, lies in $C^\infty(\mathbb{R} \to \mathbb{R})$.
- ○ …, defined by $f(x) = \ln x$ (with the domain restricted to $\mathbb{R}^+$), lies in $C^\infty(\mathbb{R}^+ \to \mathbb{R})$.

b) Which statements on differentiable manifolds are true?

- ○ Every smooth manifold has a maximal atlas.
- ○ Every C^2-manifold is also a C^1-manifold.
- ○ Every differentiable manifold is also a topological manifold.
- ○ In a C^∞-manifold, chart maps are differentiable.
- ○ Definition of a C^k-curve requires a $C^{m \geq k}$-manifold.

c) A differentiable manifold

- ○ is always a topological space.
- ○ has enough structure to constitute a vectorspace.
- ○ is a generalisation of a set.
- ○ requires the discrete topology.
- ○ features at least one set of charts that covers the whole manifold.

Exercise 2: Restricting the atlas

From topological to differentiable manifolds.

Let $(\mathbb{R}, \mathcal{O}_{\text{standard}})$ be a topological space. Let it be further equipped with an atlas $\mathcal{A} = \{(\mathbb{R}, x), (\mathbb{R}, y)\}$ where $x : \mathbb{R} \to \mathbb{R};\ a \mapsto x(a) = a$ and $y : \mathbb{R} \to \mathbb{R};\ a \mapsto y(a) = a^3$

Question: Construct the chart transition map $y \circ x^{-1} : \mathbb{R} \to \mathbb{R}$ and give its differentiability class.

Solution:

Question: Also construct the chart transition map $x \circ y^{-1} : \mathbb{R} \to \mathbb{R}$. Is $(\mathbb{R}, \mathcal{O}_{\text{standard}}, \mathcal{A})$ a differentiable manifold?

Solution:

Question: Restrict the atlas $\mathcal{A}$ to an atlas $\tilde{\mathcal{A}}$ in order to make $\left(\mathbb{R}, \mathcal{O}_{\text{standard}}, \tilde{\mathcal{A}}\right)$ into a smooth manifold.

Solution:

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Exercise 3: Soft squares on $\mathbb{R} \times \mathbb{R}$

From charts to atlases.

Let $M = \mathbb{R} \times \mathbb{R}$ equipped with the soft square topology $\mathcal{O}_{\text{ssq}}$ and an atlas $\mathcal{A} = \{(U_n, x_n)\}$, where $U_n = \{(x, y) \in \mathbb{R} \times \mathbb{R}, |\ |x| < n, |y| < n, n \in \mathbb{N}^+\}$ and

$$x_n : U_n \to x_n(U_n) \subset \mathbb{R}^2;\ (x, y) \mapsto x_n((x, y)) := \left(\frac{x+y}{2n}, \frac{x-y}{2n}\right)\ .$$

Question: Recall the definition of a chart and show that the (U_n, x_n) are indeed charts.

Solution:

Question: Show that $\mathcal{A}$ is a C^k-atlas by explicitly constructing the chart transition map. What is k?

Solution:

Question: Construct at least one other chart that would lie in the maximal extension of $\mathcal{A}$ and prove that it does.

Solution:

Short exercise 4: Undergraduate multi-dimensional analysis

A good notation and basic results for partial differentiation.

For a map $f : \mathbb{R}^d \to \mathbb{R}$ we denote by the map $\partial_i f : \mathbb{R}^d \to \mathbb{R}$ the partial derivative with respect to the i-th entry.

Question: Given a function

$$f : \mathbb{R}^3 \to \mathbb{R};\ (\alpha, \beta, \delta) \mapsto f(\alpha, \beta, \delta) := \alpha^3\beta^2 + \beta^2\delta + \delta$$

calculate the values of the following derivatives:

Solution:

- ○ $(\partial_2 f)(x, y, z) =$
- ○ $(\partial_1 f)(\square, \circ, \star) =$
- ○ $(\partial_1\partial_2 f)(a, b, c) =$
- ○ $(\partial_3^2 f)(299, 1222, 0) =$

Exercise 5: Differentiability on a manifold

How to deal with functions and curves in a chart.

Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth d-dimensional manifold. Consider a chart (U, x) of the atlas $\mathcal{A}$ together with a smooth curve $\gamma : \mathbb{R} \to U$ and a smooth function $f : U \to \mathbb{R}$ on the domain U of the chart.

Question: Draw a commutative diagram containing the chart domain, chart map, function, curve, and the respective representatives of the function and the curve in the chart.

Solution:

Question: Consider, for $d = 2$,

$$(x \circ \gamma)(\lambda) := (\cos(\lambda), \sin(\lambda)) \qquad \text{and} \qquad (f \circ x^{-1})((x, y)) := x^2 + y^2.$$

Using the chain rule, calculate

$$(f \circ \gamma)'(\lambda)$$

explicitly.

Solution: