THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

GRAVITY AND LIGHT

Tangent Spaces

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) A tangent vector to a differentiable manifold

- ○ has a length and a direction.
- ○ must not be zero.
- ○ maps a function on the manifold to the real numbers.
- ○ is of the same dimension as the manifold.
- ○ arises as the velocity to some curve through the vector's base point.

b) The tangent space T_pM to a d-dimensional differentiable manifold

- ○ is of the dimension $2 \cdot \dim M$.
- ○ can be defined at *every* point p of M.
- ○ is no real vector space, because its elements are only tangent vectors.
- ○ admits a linear bijection to the vector space $(\mathbb{R}^d, \oplus, \odot)$.
- ○ has no tangent vectors in common with a tangent space T_qM for $q \neq p$.

c) If (U, x) is a chart for a d-dimensional differentiable manifold, then

- ○ the coordinate maps $x^i \colon U \to \mathbb{R}$ with $i = 1, \dots, d$ are only continuous, not differentiable.
- ○ $(\mathrm{d}x^1)_p, \dots, (\mathrm{d}x^d)_p$ constitute a basis of T_p^*M.
- ○ for the basis $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p$ of T_pM, there is no dual basis in the dual space.
- ○ the components of the vector X with respect to the chart-induced basis are $(\mathrm{d}x^i)_p(X)$.
- ○ the expression $(\mathrm{d}x^a)_p(\left(\frac{\partial}{\partial x^a}\right)_p)$ yields the dimension of the manifold.

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Exercise 2: Virtuoso use of the symbol $\left(\frac{\partial}{\partial x^i}\right)_p$

Translating the symbol into undergraduate analysis symbols and vice versa.

Question: For a smooth function f and a chart (U, x), provide the definition of the expression

$$\left(\frac{\partial f}{\partial x^i}\right)_p .$$

Solution:

Question: Show that, for overlapping charts (U, x) and (V, y), one has

$$\left(\frac{\partial x^a}{\partial y^m}\right)_p \left(\frac{\partial y^m}{\partial x^b}\right)_p = \delta^a_b$$

for any $p \in U \cap V$.

Solution:

Question: After inserting $y^{-1} \circ y$, where y is another chart map on the same chart domain U, at an appropriate position in the definition of the left hand side of

$$\left(\frac{\partial f}{\partial x^i}\right)_p = \left(\frac{\partial y^m}{\partial x^i}\right)_p \left(\frac{\partial f}{\partial y^m}\right)_p ,$$

use the undergraduate multi-dimensional chain rule to show that it equals the right hand side.

Solution:

Question: Do the $\dim M$ many quantities defined by the left hand side of the above expression constitute the components of a tensor? If so, what are the valence and the rank of the tensor?

Solution:

Exercise 3: Transformation of vector components

Understanding the vector component transformation law the pedestrian way.

Let the topological manifold $(\mathbb{R}^2, \mathcal{O}_{\text{st.}})$ be equipped with the atlas $\mathcal{A} = \{(\mathbb{R}^2, x), (\mathbb{R}^2, y)\}$, where

$$x\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad y\colon \mathbb{R}^2 \to \mathbb{R}^2$$
$$(a,b) \mapsto (a,b) \qquad (a,b) \mapsto (a, b + a^3).$$

Question: Calculate the objects $\left(\frac{\partial x^i}{\partial y^j}\right)_p$!

Solution:

In the lectures, the velocity $v_{\gamma,p}$ of the curve at a point $p = \gamma(\lambda_0)$ has been defined by its action on a smooth function f

$$v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0).$$

By choosing a chart (U, x), inserting $x^{-1} \circ x$ at the appropriate place in this definition and employing the chain rule, you found the components of the velocity with respect to the chart

$$\dot\gamma_x^i(\lambda_0) := (x \circ \gamma)^{i\prime}(\lambda_0).$$

Now consider the curve

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2; \quad \lambda \mapsto (\lambda, -\lambda).$$

Question: Calculate the components $\dot\gamma_x^i(\lambda_0)$ and $\dot\gamma_y^i(\lambda_0)$!

Solution:

Question: With the result of the first question, how could you have obtained the components $\dot{\gamma}^i_x(\lambda_0)$ from the $\dot{\gamma}^i_y(\lambda_0)$?

Solution:

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Exercise 4: The gradient

Not the only covector undergoing an identity crisis.

Given a function f on a manifold M, the level sets of f for a constant $c \in \mathbb{R}$ are defined as

$$N_c(f) := \{p \in M \mid f(p) = c\}.$$

Question: Formulate the condition for a curve $\gamma \colon \mathbb{R} \to M$ to take values solely in one of the level sets of a function f!

Solution:

Question: Now show that the gradient of a function annihilates the velocity vector $v_{\gamma,p}$ for any such curve γ through a point p in $N_c(f)$. In other words, show that

$$(\mathrm{d}f)_p\,(v_{\gamma,p}) = 0.$$

Solution:

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Exercise 5: Is there a well-defined sum of curves?

On the dangers of defining concepts by use of charts.

Let the topological manifold $(\mathbb{R}^2, \mathcal{O}_{\text{st.}})$ be equipped with the atlas $\mathcal{A} = \{(\mathbb{R}^2, x), (\mathbb{R}^2, y)\}$, where

$$x\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad (a,b) \mapsto (a,b) \qquad\qquad y\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad (a,b) \mapsto (a, b\cdot e^a).$$

Question: Is $\mathcal{A}$ a C^∞-atlas?

Solution:

Question: On M consider the curves $\gamma\colon \mathbb{R} \to M$ and $\delta\colon \mathbb{R} \to M$, mapping

$$\gamma\colon \lambda \mapsto (\lambda, 1) \quad \text{and}$$
$$\delta\colon \lambda \mapsto (1, \lambda).$$

Without referring to any chart, can you give the sum $\gamma + \delta$ of these curves?

Solution:

Question: Calculate the representatives of both curves with respect to both charts! Illustrate the results! Where do the curves in the charts intersect?

Solution:

Recall the definition of the sum of curves with respect to a chart (U, x) from the lectures. There, for curves γ and δ meeting in the common point $\gamma(\lambda_0) = \delta(\lambda_1)$, we defined

$$\sigma_x\colon \mathbb{R} \to U, \quad \lambda \mapsto x^{-1}\left(x(\gamma(\lambda + \lambda_0)) + x(\delta(\lambda + \lambda_1)) - x(\gamma(\lambda_0))\right)$$

as the sum of γ and δ in the real world.

Question: Implement this construction with our chart $(\mathbb{R}^2, x)$ in order to determine the sum σ_x of our curves γ and δ! Draw the result in the real world.

Solution:

Question: Repeat the construction, but now using the chart $(\mathbb{R}^2, y)$ to obtain the curve σ_y. Do you get the same curve in the real world?

Solution:

Question: Show that—despite the above result—the velocity of σ_x equals the velocity of σ_y at the intersection point.

Solution: