Fields

Exercise 1: True or false

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) A bundle

- ○ is uniquely determined by its base space and its total space.
- ○ consists of a smooth manifold and an injective map to a topological manifold.
- ○ gives rise to the notion of fibres.
- ○ projection π has open preimages in the total space if fed with open sets of the base space.
- ○ allows to define a section $\sigma : M \to TM$ that has to fulfill $\sigma \circ \pi = \mathrm{id}_{TM}$.

b) The total space of a tangent bundle over a smooth manifold M

- ○ is constructed by intersecting all tangent spaces T_pM.
- ○ has double the dimension of the base space.
- ○ carries an atlas inherited from the atlas of M.
- ○ is the target space of a smooth vector field.
- ○ contains only tangent vectors.

c) A tensor field on a smooth manifold M

- ○ is a $C^\infty(M)$-linear map.
- ○ of valence (r, s) sends r covector fields and s vector fields to a smooth function.
- ○ can be added to another tensor field on the same manifold.
- ○ is a smooth function if it is of valence $(0, 0)$.
- ○ has as its components $(r + s)^{\dim M}$ many smooth functions w. r. t. a chart.

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Exercise 2: Vector fields for practitioners

Recognizing and dealing with vector fields.

Question: Let (U, x) be a chart of a smooth manifold $(M, \mathcal{O}, \mathcal{A})$. Explain why the map

$$\begin{aligned} \frac{\partial}{\partial x^i} : U &\longrightarrow TU \\ p &\mapsto \left(\frac{\partial}{\partial x^i}\right)_p \end{aligned}$$

is a vector field on U.

Solution:

Question: Show that for smooth functions f and g on M

$$\frac{\partial}{\partial x^i}(f \cdot g) = \frac{\partial f}{\partial x^i} \cdot g + f \cdot \frac{\partial g}{\partial x^i}$$

after first identifying what the different mathematical objects of this equation are and thus on which space $+$ and $\cdot$ are defined.

Solution:

Question: Expand a vector field χ on the domain of a chart (U, x) in terms of component functions $\chi^i \in C^\infty(M)$.

Solution:

Question: By application to χ of which covector field does one obtain the component functions χ^i?

Solution:

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Exercise 3: The cotangent bundle $T^*M \xrightarrow{\pi} M$

Constructing the cotangent bundle as a smooth manifold.

We consider the *cotangent bundle* total space T^*M as the disjoint union

$$T^*M := \dot{\bigcup_{p\in M}} T_p^*M$$

of all cotangent spaces and define the bundle projection map

$$\begin{aligned} \pi\colon T^*M &\longrightarrow M \\ \omega &\mapsto \text{the unique } p \text{ with } \omega \in T_p^*M \,. \end{aligned}$$

Question: Show that

$$\mathcal{O}_{T^*M} := \{\text{preim}_\pi(U) \mid U \in \mathcal{O}_M\}$$

defines a topology on T^*M.

Solution:

Question: Adapting the construction of the tangent bundle $T^*M \xrightarrow{\pi} M$, demonstrated in the lectures, construct a chart (T^*U, ξ^*_x) by defining a chart map

$$\xi^*_x : T^*U \longrightarrow \mathbb{R}^{2\dim M}$$

for each chart (U, x) of the base space M.

Solution:

Question: Find the inverse ${\xi^*_x}^{-1}$ of such a chart map!

Solution:

Question: To consider smoothness of T^*M, we need to consider chart transition maps on T^*M. Calculate the chart transition map $\xi^*_y \circ {\xi^*_x}^{-1}$.

Solution: $\xi^*_y \circ {\xi^*_x}^{-1}\left(\alpha^1, \dots, \alpha^{\dim M}, \gamma_1, \dots, \gamma_{\dim M}\right) =$

Question: Why is this chart transition map smooth?

Solution: