THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

GRAVITY AND LIGHT

Connections

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) A covariant derivative operator ∇_X

- ○ is $C^\infty(M)$-linear in X.
- ○ cannot be applied to $(0, 0)$-tensor fields.
- ○ acts simply as the partial derivative if the manifold is flat.
- ○ applied to a smooth function is the gradient of this function.
- ○ is uniquely fixed by choosing $(\dim M)^3$ many connection coefficient functions of an atlas.

b) Which statements on connections are true?

- ○ A connection maps a vector field and a (p, q)-tensor field to the real numbers.
- ○ A connection satisfies the Leibniz rule.
- ○ Connection coefficients vanishing in a chart, vanish in the overlap with another chart.
- ○ The connection coefficients may be defined by ∇ and the chart-induced basis fields.
- ○ A torsion-free connection has coefficient functions satisfying $\Gamma^k{}_{[ab]} = 0$ in all charts.

Connections

Exercise 2: Practical rules for how ∇ acts

How does a covariant derivative act on various tensor fields?

What is the result of the following applications of a torsion-free covariant derivative?

- $\nabla_X f =$
- $(\nabla_X Y)^a =$
- $(\nabla_X \omega)_a =$
- $(\nabla_m T)^a{}_{bc} =$
- $(\nabla_{[m} A)_{n]} =$
- $(\nabla_{[m} \omega)_{nr]} =$

Connections

Exercise 3: Connection coefficients

Vanishing connection coefficients - really vanishing?

Question: Recall how the connection coefficient functions in a chart $(U \cap V, y)$ are related to those in a chart $(U \cap V, x)$?

Solution:

Question: Which class of chart transition maps make the connection coefficient functions appear to transform like a tensor?

Solution:

Question: Let $(M, \mathcal{O}, \mathcal{A}, \nabla)$ be the flat plane. Consider two charts that both cover the upper-half plane, that is all points $(a, b) \in \mathbb{R}^2$ with $b > 0$, one representing the familiar cartesian coordinates and the familiar polar coordinates on there.
We already know that the chart transition map from cartesian to polar coordinates is given by

$$y \circ x^{-1}(a, b) = \left(\sqrt{a^2 + b^2}, \arccos\left(\frac{a}{\sqrt{a^2 + b^2}}\right)\right),$$

while the inverse transition map from polar coordinates to cartesian ones is

$$x \circ y^{-1}(r, \varphi) = (r \cos\varphi, r \sin\varphi) \qquad \text{for } r \in \mathbb{R}^+ \text{ and } \varphi \in (0, \pi).$$

Starting from the assumption of vanishing connection coefficient functions in the cartesian chart, calculate the connection coefficients ${\Gamma^a}_{bc\,(y)}$ with respect to the polar coordinate chart!

Solution:

Connections

Exercise 4: Why, on functions, ∇ reduces to d

Checking the compatibility of the first axiom for a connection with the remaining axioms.

Question: Recall how ∇_X acts on a function $h \in \mathcal{C}^\infty(M)$.

Solution:

$$\nabla_X h =$$

Question: Complete the sketch of proof below—for the scaling part of the $\mathcal{C}^\infty(M)$-linearity—by

a) indicating over each S-multiplication on which space it is defined,

b) giving sufficient justification for each equal sign.

Solution:

$$\nabla_{g\cdot X}\, h \quad = \quad (g \cdot X)h \quad = \quad g \cdot Xh \quad = \quad g \cdot \nabla_X h$$

Question: Provide analogous proofs for the other axioms.

Solution:

$\nabla_{X+Y}\, h =$

$\nabla_X(h + g) =$

$\nabla_X(f \cdot g) =$