THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

GRAVITY AND LIGHT

Parallel transport & Curvature

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) It is correct that

- ○ a parallel vector field is certainly a parallely transported vector field.
- ○ on a flat manifold, the autoparallel equation for a curve γ in any chart (U, x) is $\ddot{\gamma}^i_{(x)}(\lambda) = 0$.
- ○ Newton's second law formulated as $F^i_{(x)}(\gamma(\lambda)) = m\ddot{\gamma}^i_{(x)}(\lambda)$ cannot be correct.
- ○ on a flat manifold, parallel transport around a closed curve results in the same vector.
- ○ the introduction of torsion does not change the autoparallel equation.

b) The Riemann curvature

- ○ is a $(3, 1)$-tensor field.
- ○ is anti-symmetric in its last two slots.
- ○ quantifies the failure of second covariant derviatives to satisfy a Schwarz rule.
- ○ has $d^3(d-1)/2$ independent components.
- ○ obstructs the existence of coordinates with respect to which the Γs vanish.

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Exercise 2: Where connection coefficients appear

Some heavily used expressions in terms of the Γ*s that should be committed to memory.*

Question: Recall the autoparallel equation for a curve γ in both

a) chart-free formulation,

b) formulation with respect to a chart (U, x).

Solution:

Question: Determine the coefficients of the Riemann tensor with respect to a chart (U, x) in terms of the connection coefficient functions.

Solution:

Question: Does $\mathrm{Ric}(X, Y) := \mathrm{Riem}^m{}_{amb} X^a Y^b$ define a $(0, 2)$-tensor?

Solution:

Question: Does a one-dimensional manifold with connection have curvature? Why?

Solution:

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Exercise 3: The round sphere

Explicit calculation of the Riemann curvature of the round sphere

Question: Consider the sphere $(S^2, \mathcal{O}, \mathcal{A})$ and a chart $(U, x) \in \mathcal{A}$ where the only non-vanishing connection coefficients are given by

$$\Gamma^1{}_{22}\,(x^{-1}(\theta,\varphi)) = -\sin\theta\cos\theta \quad \text{and} \quad \Gamma^2{}_{12}\,(x^{-1}(\theta,\varphi)) = \Gamma^2{}_{21}\,(x^{-1}(\theta,\varphi)) = \cot\theta$$

for $\varphi \in (0, 2\pi)$ and $\theta \in (0, \pi)$.

Calculate the component functions $\mathrm{Riem}^1{}_{212}$ and $\mathrm{Riem}^1{}_{112}$ of the Riemann curvature on the sphere! Give a list of the remaining component functions.

Solution:

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Exercise 4: How not to define parallel transport

Why comparing components of vectors in different tangent spaces doesn't work.

Question: Hänschen defines two vectors $X \in T_pM$ and $Y \in T_qM$ as parallel if

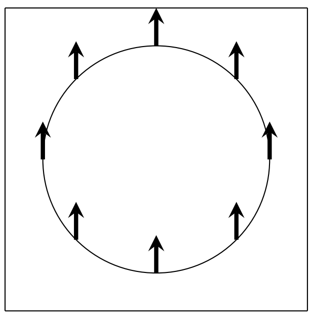
$$X^i_{(x)} = Y^i_{(x)}$$

with respect to some chart (U, x) whose domain U contains both p and q.

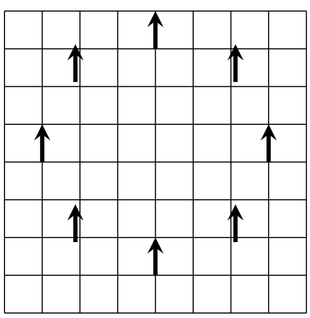
Prove that this notion of parallelity is ill-defined!

Solution:

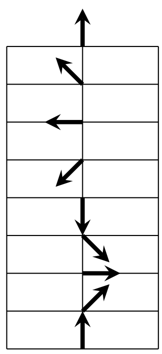
Question: Explain the full extent of the disaster by discussing the below two chart representations of the same eight vectors that are parallel (according to the notion introduced in the lecture) in the flat plane. How does one usually refer to the charts (U, x) and (V, y)?



(a) The Real World $\mathbb{R}^2$



(b) Chart (U, x)



(c) Chart (V, y)

Solution: