Metric Manifolds

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) The Levi-Civita connection
- ○ is torsion-free.
- ○ is called metric-compatible since it satisfies $\nabla_X g = g$.
- ○ is the unique affine connection that can be constructed from the data of a metric manifold.
- ○ gives rise to connection coefficient functions with $\Gamma^a{}_{(bc)} = 0$ in any chart.
- ○ arises by requiring that the autoparallels are the curves of stationary length.

b) Which statements about metrics on a d-dimensional smooth manifold are correct?
- ○ A metric with signature (d, d) is called a Riemannian metric.
- ○ A Lorentzian metric has signature $(1, d-1)$.
- ○ On a Lorentzian manifold, vectors $X \in T_pM$ are called g-null vectors if $g(X, X) = 0$.
- ○ A metric provides an inner product on each tangent space T_pM.
- ○ An inverse metric fed with a covector yields a real number.

c) Which statements about geodesics and the length of a curve are correct?
- ○ A Riemannian metric gives rise to a well-defined notion of length of a curve.
- ○ The length of a curve is obviously not invariant under reparametrization of this curve.
- ○ A geodesic is a curve of minimal length.
- ○ The condition for a curve to be of stationary length is that it be a solution of the Euler-Lagrange equation for the Lagrangian $\mathcal{L}(X) = \sqrt{g(X, X)}$.
- ○ A geodesic on a metric manifold satisfies the autoparallel equation.

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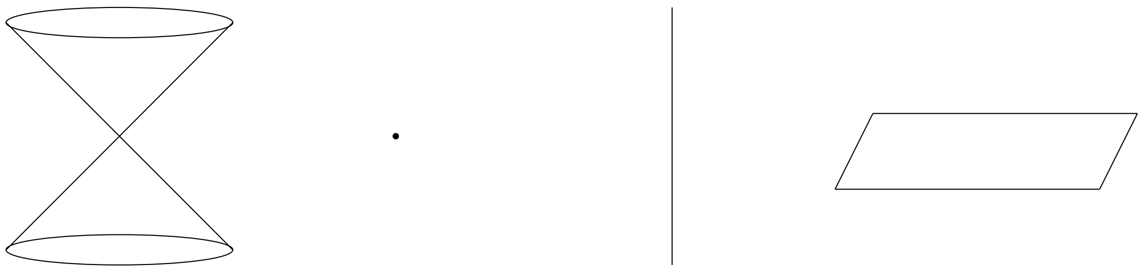
Exercise 2: Recognizing and dealing with different signatures

Other than $(1,1)$-tensors, $(0,2)$-tensors do not have eigenvalues, but signature.

Let g be a symmetric $(0,2)$-tensor over some vector space.

Question: You are given the following four sets of g-null vectors in the vector space: the surface of a double cone centered in the origin, a point in the origin, a straight line through the origin and a plane through the origin of the vector space.

Determine the possible signatures of the respective g in each of the four cases.



(a) (b) (c) (d)

Solution:

(a)

(b)

(c)

(d)

Question: For each signature you found, indicate the set of vectors of positive length in the drawings.

Solution:

Exercise 3: Levi-Civita Connection

For vanishing torsion, the Levi-Civita connection is already uniquely determined by the requirement $\nabla g = 0$.

Suppose we have a torsion-free and metric-compatible connection, i.e., vanishing rank-three tensor fields

$$T = 0 \qquad \text{and} \qquad \nabla g = 0.$$

Question: Recall what $T = 0$ implies for the connection coefficient functions with respect to a chart.

Solution:

Question: Expand in terms of connection coefficient functions.

Solution:

(I) $(\nabla_a g)_{bc} =$

(II) $(\nabla_b g)_{ca} =$

(III) $(\nabla_c g)_{ab} =$

Question: By adding and/or subtracting (I), (II) and (III) in a clever way, obtain

$$\Gamma^a{}_{bc} = \frac{1}{2}\left(g^{-1}\right)^{am}\left(\frac{\partial}{\partial x^b} g_{mc} + \frac{\partial}{\partial x^c} g_{mb} - \frac{\partial}{\partial x^m} g_{bc}\right)$$

and conclude that the conditions $\nabla g = 0$ and $T = 0$ uniquely determine the connection coefficient functions in terms of the metric.

Solution:

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Exercise 4: Massaging the length functional

Modifications of the length functional that simplify calculations but do not change results.

Question: Let $\gamma : (0,1) \longrightarrow M$ be a smooth curve on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$. Now consider a second curve $\tilde{\gamma} : (0,1) \longrightarrow M$ defined by

$$\tilde{\gamma}(\lambda) := \gamma(\sigma(\lambda)),$$

where $\sigma : (0,1) \longrightarrow (0,1)$ is an increasing bijective smooth function.

Show that the length of both curves is the same:

$$L[\tilde{\gamma}] = L[\gamma]$$

Solution:

Question: Show that the Euler-Lagrange equations for a Lagrangian $\mathcal{T}$ have precisely the same solutions as the Euler-Lagrange equations for the Lagrangian $\mathcal{L} := \sqrt{\mathcal{T}}$, if of the latter one only selects those solutions that satisfy the condition $\mathcal{T} = 1$ on their parametrization.

Solution:

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Exercise 5: A practical way to quickly determine Christoffel symbols

In a concrete case, rederiving the Euler-Lagrange equations is quicker than using the general formula.

Question: Derive the geodesic equation for the two-dimensional round sphere of radius R, whose metric in some chart (U, x) is given by

$$g_{ab}\left(x^{-1}(\vartheta, \phi)\right) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \vartheta \end{pmatrix},$$

via a convenient Euler-Lagrange equation. In order to lighten the notation, you may define

$$\vartheta(\lambda) := \left(x^1 \circ \gamma\right)(\lambda) \quad \text{and} \quad \phi(\lambda) := \left(x^2 \circ \gamma\right)(\lambda).$$

Solution:

Question: Read off the metric-induced connection coefficient functions for the round sphere.

Solution:

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Exercise 6: Properties of the Riemann-Christoffel tensor

Various algebraic symmetries that the plain Riemann curvature does not feature.

Question: Show that the chart-induced basis fields act on the coefficient functions as

$$\frac{\partial}{\partial x^c}\left(g^{-1}\right)^{ab} = -\left(g^{-1}\right)^{ar}\left(g^{-1}\right)^{bs}\frac{\partial}{\partial x^c}g_{rs}.$$

Solution:

Question: Use normal coordinates to find an expression for the Riemann-Christoffel tensor

$$R_{abcd} := g_{ak}\,R^k{}_{bcd}$$

at a given point p in terms of the metric g_{ab} and its first and second derivatives at that very point.

Solution:

Question: Show—in normal coordinates—that $R_{abcd} = -R_{bacd}$.

Solution:

Question: Similarly, show that $R_{abcd} = R_{cdab}$.

Solution:

Question: Show that $R_{a[bcd]} = 0$ for the Riemann-Christoffel tensor.

Solution: