Symmetry

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) Which statements about push-forward and pull-back are correct?

- ○ The push-forward maps a vector field to a vector field.
- ○ The pull-back of a covector yields a covector.
- ○ One can induce a metric on a smoothly embedded manifold by pulling the metric back from the ambient manifold.
- ○ Push-forward and pull-back are linear maps.
- ○ The push-forward is a $(1,1)$-tensor.
- ○ One can only pull back a $(1,1)$-tensor along a bijective map.

b) Which statements are correct?

- ○ A Lie subalgebra $L \subset \Gamma(TM)$ is a symmetry of a metric tensor field g if the pull-back of g along the flow of any element of L reproduces g.
- ○ The Lie derivative of a vector field Y with respect to a vector field X is the commutator $[X, Y]$.
- ○ A Lie subalgebra $L \subset \Gamma(TM)$ is a symmetry of a metric tensor field g if the Lie derivative of g with respect to every $X \in L$ vanishes.
- ○ The Lie derivative acts on a function as the covariant derivative does.

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Exercise 2: Pull-back and push-forward

Formulae for practical men and women.

Question: Consider a smooth map $\phi\colon M \to N$ between two differentiable manifolds. Show that for a function $f \in C^\infty(N)$, the pull-back of the gradient of f is the same as the gradient of the pull-back of f, i.e.

$$\phi^*(\mathrm{d}f) = \mathrm{d}(\phi^* f).$$

Solution:

Question: The push-forward $\phi_*\colon TM \xrightarrow{\sim} TN$ is a linear map between tangent bundles. Calculate its component functions

$$\phi_*{}^a{}_b := (\mathrm{d}y^a)(\phi_*(\frac{\partial}{\partial x^b}))$$

with respect to charts $(U \subset M, x)$ and $(V \subset N, y)$!

Solution:

Question: Show that the component functions of the pull back $\phi^* g$ of a metric tensor field are obtained from the component functions of g by

$$(\phi^* g)_{ab}(p) = \left(\frac{\partial (y \circ \phi)^m}{\partial x^a}\right)_p \left(\frac{\partial (y \circ \phi)^n}{\partial x^b}\right)_p g_{mn}(\phi(p)).$$

Solution:

Exercise 3: Lie derivative—the pedestrian way

The Lie derivative as a tool for investigating symmetries.

Question: Consider the smooth embedding $\iota\colon S^2 \to \mathbb{R}^3$ of $(S^2, \mathcal{O}, \mathcal{A})$ into $(\mathbb{R}^3, \mathcal{O}_{\text{st}}, \mathcal{B})$, which for the familiar spherical chart $(U, x) \in \mathcal{A}$ and $(\mathbb{R}^3, y = \text{id}_{\mathbb{R}^3}) \in \mathcal{B}$ is given by

$$y \circ \iota \circ x^{-1}\colon (\vartheta, \varphi) \mapsto (a \cos\varphi \sin\vartheta, b \sin\varphi \sin\vartheta, c \cos\vartheta),$$

where a, b and c are positive real numbers. What can you say about the shape of $\iota(S^2)$?

Solution:

Question: Now assume that $(\mathbb{R}^3, \mathcal{O}_{\text{st}}, \mathcal{B})$ is additionally equipped with the Euclidean metric g, whose component functions with respect to the chart $(\mathbb{R}^3, y)$ are given by

$$g_{ab}(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ab} \quad \text{for any } p \in U.$$

Write down the component functions of $g^{\text{ellipsoid}} := \iota^* g$ with respect to the chart (U, x)!

Solution:

Question: For convenience, denote by (ϑ, φ) the coordinate functions (x^1, x^2). Check that the vector fields

$$\begin{aligned} X_1(p) &= -\sin\varphi(p)\left(\frac{\partial}{\partial\vartheta}\right)_p - \cot\vartheta(p)\cos\varphi(p)\left(\frac{\partial}{\partial\varphi}\right)_p \\ X_2(p) &= \cos\varphi(p)\left(\frac{\partial}{\partial\vartheta}\right)_p - \cot\vartheta(p)\sin\varphi(p)\left(\frac{\partial}{\partial\varphi}\right)_p \\ X_3(p) &= \left(\frac{\partial}{\partial\varphi}\right)_p \end{aligned}$$

constitute a Lie subalgebra of $(\Gamma(TS^2), [\cdot, \cdot])$ and determine its structure constants!

Solution:

Question: Calculate the integral curve of X_3 through the point $p = x^{-1}(\vartheta_0, \varphi_0)$, i.e. the curve γ_p satisfying

$$\gamma_p(0) = p \qquad \text{and} \qquad v_{\gamma_p, \gamma_p(\lambda)} = (X_3)_{\gamma_p(\lambda)}$$

in the chart (U, x)!

Solution:

Question: The integral curves γ_p give rise to a one-parameter family of smooth maps $h_\lambda^{X_3} : S^2 \to S^2$. Calculate the pull-back

$$(h_\lambda^{X_3})^* g^{\text{ellipsoid}}$$

of the metric on S^2. What can you conclude for the Lie derivative $\mathcal{L}_{X_3} g$?

Solution: