THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

# GRAVITY AND LIGHT

## Schwarzschild Spacetime

## Exercise 1: Geodesics in a Schwarzschild spacetime

*Deriving the geodesic equations for the Schwarzschild metric and solving them*

Consider a manifold equipped with the Schwarzschild metric

$$g_{ab} = \begin{pmatrix} 1-\frac{2GM}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} .$$

In the following, we will use the following notation for some chart $(U, x)$ that may cover an arbitrarily large part of the manifold. For a curve $\gamma : \mathbb{R} \to U$, we have

$$t(\lambda) := (x^0 \circ \gamma)(\lambda) = \gamma^0_{(x)} , \qquad r(\lambda) := (x^1 \circ \gamma)(\lambda) = \gamma^1_{(x)} ,$$
$$\theta(\lambda) := (x^2 \circ \gamma)(\lambda) = \gamma^2_{(x)} , \qquad \varphi(\lambda) := (x^3 \circ \gamma)(\lambda) = \gamma^3_{(x)} .$$

**Question:** Write down the Lagrangian $L = g_{ab}\dot\gamma^a_{(x)}\dot\gamma^b_{(x)}$!

**Solution:**

**Question:** The Lagrangian can be parametrized by $\lambda$ in such a way that $L = 1$ for all $\lambda$. This will prove useful later on.
There are four Euler-Lagrange equations for this Lagrangian. Derive the one with respect to the function $t(\lambda)$!

**Solution:**

**Question:** Show that the Lie derivative of $g$ with respect to the vector field $K_t := \frac{\partial}{\partial t}$ vanishes. What does this mean?

**Solution:**

**Question:** The exact form of the conserved quantity is given by $(K_t)_a (x^a)'(\lambda) = \text{const.}$ (without proof). Derive an expression for the quantity $t'(\lambda)$ appearing in the Lagrangian!

**Solution:**

**Question:** Moreover, we find so called "spherical symmetry", that is, the Lie derivative of $g$ with respect to the already known vector fields

$$\begin{aligned} X_1 &= \sin\varphi \frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} \\ X_2 &= \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} \\ X_3 &= \frac{\partial}{\partial\varphi} \end{aligned}$$

vanishes. Which physical quantity is conserved by this symmetry?

**Solution:**

**Question:** Due to $X_1$ and $X_2$ (without proof), one can fix the motion to a plane of constant $\theta = \frac{\pi}{2}$. How can you derive an expression for the remaining term $\varphi'(\lambda)$?

**Solution:**

**Question:** Use the fact that $L = 1$ on the parametrization. Insert the previously obtained results and take all terms not containing $E$ to one side!

**Solution:**

**Question:** Can you interpret the terms appearing in this expression?

## Schwarzschild Spacetime

### Exercise 2: Gravitational redshift

*Light propagation in a Schwarzschild spacetime*

Consider a spacetime equipped with Schwarzschild metric as well as two observers 1 and 2 at rest in their respective system of reference ($\dot{r} = 0, \dot{\theta} = 0, \dot{\varphi} = 0$). The observers sit at the same $\theta$ and $\phi$, while $r_1 < r_2$.

**Question:** Derive an expression for $t'(\lambda)$ using the Lagrangian from the previous exercise!

**Solution:**

**Question:** Observer 1 emits photons that observer 2 detects. The gap between two photon emissions is $\Delta\lambda_1$. Find the gap $\Delta\lambda_2$ seen by observer 2!

**Solution:**

**Question:** Consider the ratio of the frequencies $\frac{\omega_2}{\omega_1}$. What happens for observer 2 being approximately at infinity? What happens when sending $r_1$ to the Schwarzschild radius $r_s = 2GM$?