THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

GRAVITY AND LIGHT

Relativistic Spacetime, Matter and Gravitation

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

a) Relativistic spacetime

- ○ has its topology always completely determined by the Einstein equations.
- ○ admits a smooth vector field T satisfying $g(T, T) > 0$.
- ○ is equipped with an oriented atlas.
- ○ has dimension 4.
- ○ features a connection which does not necessarily arise from a metric.

b) Which statements are correct?

- ○ An observer can sit on a massless particle.
- ○ The eigentime of an observer is just the length of her curve.
- ○ The worldline γ of a particle satisfies $g(T, v_\gamma) > 0$ for the time orientation T.
- ○ Lorentz transformations relate different observer frames at different points.

c) It is not correct that

- ○ any matter field is a function on spacetime.
- ○ the energy momentum tensor of a perfect fluid has the form $T^{ab} = (\rho + p)u^a u^b - pg^{ab}$.
- ○ the Einstein-Hilbert action is linear in the curvature.
- ○ the matter action of electrodynamics is linear in the field tensor.

Exercise 2: Lorentz force law

Extracting familiar formulae in an observer's frame from objective spacetime quantities.

Question: Recall the precise definition of an observer.

Solution:

Question: Recall from the lecture that for a particle coupling to the electromagnetic potential, we have

$$m\left(\nabla_{\vartheta_\gamma}\vartheta_\gamma\right)^a = q\,F^a_{\ b}\,\vartheta_\gamma{}^b\ ,$$

where ϑ_γ is the velocity of the particle with mass m and charge q.

Now "1+3"-decompose this equation in components with respect to the frame of an observer.

Solution:

Question: Using the definitions $E_\alpha := F_{\alpha 0}$ for the electric field and $B^\alpha := \frac{1}{2}\varepsilon^{\alpha\rho\sigma} F_{\rho\sigma}$ for the magnetic field seen by an observer, bring the right hand side of the above equation to the familiar form of the Lorentz force law for a particle of charge q and spatial velocity

$$\mathrm{v} := \frac{\epsilon^\alpha(\vartheta_\delta)}{\epsilon^0(\vartheta_\delta)} e_\alpha \qquad (\alpha = 1,2,3 \text{ and careful: denominator was forgotten in the lectures})$$

that the observer detects for the particle.

Hint: $(a \times b)^\alpha = g^{\alpha\mu} \varepsilon_{\mu\rho\sigma} a^\rho b^\sigma$, $\varepsilon_{123} = 1$ and $\varepsilon^{123} = 1$.

Solution:

Exercise 3: Which curvature can feature in Einstein's equations?

The divergence of the Einstein tensor vanishes, like that of an energy-momentum tensor.

Question: Show that for a torsion-free connection, the so-called differential Bianchi identity

$$(\nabla_A R)(\omega, Z, B, C) + (\nabla_B R)(\omega, Z, C, A) + (\nabla_C R)(\omega, Z, A, B) = 0$$

holds. Furthermore, the Jacobi identity for the commutator might be helpful.

Hint: Start by rewriting the first term only by repeated use of the Leibniz rule and one-time employment of the definition of the Riemann tensor. From this result, generate the second and third terms by mere cyclic substitution of the appropriate vectors. The rest is systematic and disciplined elimination of terms.

Solution:

Question: The above component-free version can be equivalently written as

$$R^{w}{}_{zab;c} + R^{w}{}_{zbc;a} + R^{w}{}_{zca;b} = 0.$$

Using this result, show that by appropriate contractions one obtains

$$(\nabla_a G)^{ab} = 0.$$

Solution: