Cosmology

Exercise 1: True or false

These basic questions are designed to spark discussion and as a self-test.

a) Which statements about maximally symmetric spaces are true?

- ○ Homogeneous and isotropic 3-manifolds are maximally symmetric manifolds.
- ○ A 5-dimensional manifold can have maximally 10 linearly independent Killing vector fields.
- ○ Riemann curvature on a maximally symmetric manifold contains exactly one derivative of the metric.
- ○ A maximally symmetric 3-manifold cannot be flat.
- ○ A maximally symmetric space has constant sectional curvatures.

b) Which statements are correct?

- ○ Perfect fluid matter has constant pressure and constant density over the whole spacetime.
- ○ Homogeneous and isotropic dust and radiation can be modelled as a perfect fluid.
- ○ The Friedmann equations are the Einstein equations for a spherically symmetric vacuum.
- ○ The FRW metric with $\kappa = 1$ is the metric of a hyperbolic 3-manifold.
- ○ All matter densities $\rho(t)$ are proportional to t^{-2}.

Exercise 2: Killing's Equation

Killing vectors lie-derivatively kill their metric.

Question: Recall the condition for a vector field K to be a Killing vector field. Give an interpretation of this condition in terms of the flow h^K associated with that Killing vector field.

Solution:

Question: Show that a vector field K is Killing if, and only if,

$$(\nabla_a K)_b + (\nabla_b K)_a = 0\,.$$

Solution:

Exercise 3: Age of the Universe ...

... depending on today's expansion rate and speculations on its matter contents.

The energy-momentum tensor for a perfect fluid is

$$T^{ab} := [\rho(t) + p(t)]\, u^a u^b + p(t) g^{ab}\,,$$

where $u^a = (1,0,0,0)^a$ are the component functions of a smooth vector field and g_{ab} those of an FRW metric with respect to the coordinate chart (t, r, ϑ, ϕ) employed in the lectures.

Question: Derive the conservation equation

$$\dot{\rho}(t) = -3\frac{\dot{a}}{a}\left(\rho(t) + p(t)\right)$$

by evaluating the condition

$$(\nabla_a T)^{ab}\, u_b = 0$$

which follows from the Einstein equations by virtue of differential Bianchi identity.

Solution:

Question: For $p(t) = \omega\rho(t)$, solve the conservation equation above for ρ and use the Friedmann equation with vanishing spatial curvature,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\,,$$

to derive an autonomous differential equation for a.

Solution:

Question: Show that

$$a(t) = C \cdot t^{\alpha} \qquad C = \text{const}$$

solves the autonomous differential equation for the scale factor a if $\omega \neq -1$ and a suitably chosen α.

Solution:

Question: Use the result of the previous question to write down an equation for $H(t) := \frac{\dot{a}}{a}$ and estimate the age of a universe only filled with dust for today's value of the Hubble constant being given by $\frac{1}{H_0} \approx 13 \times 10^9\, a$. Repeat the calculation for a universe containing only radiation.

Solution:

Question: Consider a universe filled with only one type of matter characterized by a linear equation of state with constant ω. For which values of the latter is the expansion of the universe accelerating.

Solution: