THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON

GRAVITY AND LIGHT

Perturbation Theory

Exercise 1: True or false?

These basic questions are designed to spark discussion and as a self-test.

Tick the correct statements, but not the incorrect ones!

- ○ Changes of coordinates are generated by vector fields.
- ○ Tensor perturbations have 3 independent polarizations.
- ○ All perturbations δg_{ab} are physical in the sense that they have observable consequences.
- ○ The Lense-Thirring effect is due to a tensor perturbation of the background.
- ○ A perturbation does not necessitate a δT^{ab}.
- ○ The decomposition of δg_{ab} into scalar, vector, and tensor perturbations depends on the choice of coordinates on the background.
- ○ The variation $\delta L[\gamma]$ of the length of a curve vanishes for vector perturbations.

Exercise 2: Schwarzschild as a perturbation of Minkowski spacetime

How do point particles change Minkowski spacetime?

Question: Consider an uncharged point particle of mass m on a worldline γ, which is described by the action

$$S_{\mathrm{PP}}[g, \gamma] = \frac{m}{2} \int \mathrm{d}\lambda \, g(v_\gamma, v_\gamma)$$

and show that the energy-stress tensor takes the form

$$T^{ab}_{\mathrm{PP}}(x^{-1}(\alpha)) = \frac{m}{\sqrt{-g}} \, \dot{\gamma}^a(\lambda) \dot{\gamma}^b(\lambda) \delta^{(3)}_{\vec{\gamma}(\lambda)}(\vec{\alpha}) \Big|_{\gamma^0(\lambda) = \alpha^0}$$

in a chart (U, x) with $\dot{\gamma}^0 = 1$.

Solution:

Question: Now let the particle remain at the origin of the standard chart (t, x^1, x^2, x^3) for the Minkowski spacetime, i.e. $\dot{\gamma}^a = (1, 0, 0, 0)^a$. Write down the components of its energy-momentum tensor!

Solution:

Question: T_{PP} can be viewed as small perturbation to the vacuum Einstein Equations. Why is that and what kind of perturbation is it? Determine the resulting perturbation of the flat metric η with the help of the handout from the lectures! You should obtain

$$\delta g_{00} = -2C, \quad \delta g_{\alpha\beta} = -2C\gamma_{\alpha\beta}$$

with

$$\Delta C = 4\pi G (T_{\text{PP}})_{00}.$$

Solution:

Question: Solve for C and write down the perturbed metric $\eta + \delta g$!

Solution:

Question: For the sake of this exercise, pretend you are Karl Schwarzschild and you just derived in the year 1915—one month after Einstein's publication on General Relativity—from certain symmetry assumptions the beautiful metric

$$g = \left(1 - \frac{a}{r}\right) \mathrm{d}t \otimes \mathrm{d}t - \left(1 - \frac{a}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r - r^2[\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \, \mathrm{d}\phi \otimes \mathrm{d}\phi],$$

which you believe should be the metric for a central body of mass m sitting at the point $r = 0$.

Verify your feeling by making use of the previous result! Can you determine a value for a?

Solution: