

Commented demonstration of the invertibility of the Fourier transform on Schwartz space. Reworked from:

- F. Schuller QM18 - Fourier Operator;
- Corresponding lectures notes (Simon Rea & Richie Dadhley)

**Definition 1.** The Fourier operator is the linear map  $\mathcal{F} : S(\mathbb{R}^d) \rightarrow S(\mathbb{R}^d)$  defined for  $x \in \mathbb{R}^d$  by:

$$(\mathcal{F}(f))(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ixy) f(y) d^d y$$

**Result 1.** A well-known integration result, recalled in the lecture:

$$\int_{\mathbb{R}} \exp(-\sigma x^2) dx = \sqrt{\frac{\pi}{\sigma}}$$

Which generalizes to  $\mathbb{R}^d$  to:

$$\int_{\mathbb{R}^d} \exp(-\sigma x^2) d^d x = \left(\frac{\pi}{\sigma}\right)^{d/2}$$

**Lemma 1.** Let  $x \in \mathbb{R}^d$  and  $z \in \mathbb{C}$ , such that  $\Re(z) > 0$ . Then:

$$(\mathcal{F}(x \mapsto \exp(-\frac{z}{2}x^2)))(p) = \frac{1}{z^{d/2}} \exp(-\frac{1}{2z}p^2)$$

**Theorem 1.**  $\mathcal{F} : S(\mathbb{R}^d) \rightarrow S(\mathbb{R}^d)$  is invertible and:

$$(\mathcal{F}^{-1}(g))(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(ipx) g(p) d^d p$$

*Proof.*

$$\begin{aligned}
(\mathcal{F}^{-1}(\mathcal{F}(f)))(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(ipx)(\mathcal{F}(f))(p)d^d p \\
&\quad \text{(executing the proposed } \mathcal{F}^{-1}\text{)} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left(\lim_{\epsilon \rightarrow 0} \exp\left(-\frac{\epsilon}{2}p^2\right)\right)}_{=1} \exp(ipx)(\mathcal{F}(f))(p)d^d p \\
&\quad \text{(regulator)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2}p^2\right) \exp(ipx)(\mathcal{F}(f))(p)d^d p \\
&\quad \text{(Lebesgue dominated convergence)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2}p^2\right) \exp(ipx) \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ipy)f(y)d^d y\right) d^d p \\
&\quad \text{(executing } (\mathcal{F}(f))(p)\text{)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2}p^2\right) \underbrace{\exp(ipx) \exp(-ipy)}_{\exp(-ip(y-x))} d^d p\right) f(y)d^d y \\
&\quad \text{(Fubini)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2}p^2\right) \exp(-ipz)d^d p\right)}_{=:(\mathcal{F}(p \mapsto -\exp(-\frac{\epsilon}{2}p^2)))(z)} f(z+x)d^d z \\
&\quad \text{(change of variable } z \leftarrow y-x; d^d z \leftarrow d^d y\text{)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left((\mathcal{F}(p \mapsto -\exp(-\frac{\epsilon}{2}p^2)))(z)\right) f(z+x)d^d z \\
&\quad \text{(Fourier operator definition)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{\epsilon^{d/2}} \exp\left(-\frac{1}{2}\left(\frac{z}{\epsilon^{1/2}}\right)^2\right) f(z+x)d^d z \\
&\quad \text{(previous lemma, with } z \leftarrow \epsilon, p \leftarrow z\text{)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\epsilon^{d/2}}{\epsilon^{d/2}} \exp\left(-\frac{t^2}{2}\right) f(t\epsilon^{1/2}+x)d^d t \\
&\quad \text{(change of variable: } t \leftarrow z/\epsilon^{1/2}, d^d t \leftarrow d^d z/\epsilon^{d/2}\text{)} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{t^2}{2}\right) \underbrace{\lim_{\epsilon \rightarrow 0} f(t\epsilon^{1/2}+x)}_{=f(x)} d^d t \\
&\quad \text{(dominated convergence)} \\
&= f(x) \underbrace{\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}t^2\right) d^d t}_{=(2\pi)^{d/2}} \\
&\quad \text{(previous well-known integration result, generalized)} \\
&= f(x) \\
&\Rightarrow \boxed{\mathcal{F}^{-1} \circ \mathcal{F} = \text{id}_{S(\mathbb{R}^d)}}
\end{aligned}$$

We've proven that the proposed  $\mathcal{F}^{-1}$  is the left inverse of  $\mathcal{F}$ ; it remains to prove it is its right inverse. But the process is very similar: we use the same regulator, dominated convergence to swap the limit and the integral, and Fubini to swap the integration order.

Hopefully, without any typos:

$$\begin{aligned}
(\mathcal{F}(\mathcal{F}^{-1}(g)))(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ixy) (\mathcal{F}^{-1}(g))(y) d^d y \\
&\quad \text{(executing } \mathcal{F} \text{)} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left( \lim_{\epsilon \rightarrow 0} \exp\left(-\frac{\epsilon}{2} y^2\right) \right)}_{=1} \exp(-ixy) (\mathcal{F}^{-1}(g))(y) d^d y \\
&\quad \text{(regulator)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2} y^2\right) \exp(-ixy) (\mathcal{F}^{-1}(g))(y) d^d y \\
&\quad \text{(Lebesgue dominated convergence)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2} y^2\right) \exp(-ixy) \left( \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(ipy) g(p) d^d p \right) d^d y \\
&\quad \text{(executing the proposed } (\mathcal{F}^{-1}(g))(y) \text{)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left( \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2} y^2\right) \underbrace{\exp(ipy) \exp(-ixy)}_{\exp(-iy(x-p))} d^d y \right) g(p) d^d p \\
&\quad \text{(Fubini)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left( \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\epsilon}{2} y^2\right) \exp(-iyt) d^d y \right)}_{=:(\mathcal{F}(y \mapsto -\exp(-\frac{\epsilon}{2} y^2)))(t)} g(x-t) d^d t \\
&\quad \text{(change of variable } t \leftarrow x-p; d^d t \leftarrow d^d p \text{)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathcal{F}(y \mapsto -\exp(-\frac{\epsilon}{2} y^2))(t) g(x-t) d^d t \\
&\quad \text{(Fourier operator definition)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{\epsilon^{d/2}} \exp\left(-\frac{1}{2} \left(\frac{t}{\epsilon^{1/2}}\right)^2\right) g(x-t) d^d t \\
&\quad \text{(previous lemma, with } z \leftarrow \epsilon, p \leftarrow t \text{)} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\epsilon^{d/2}}{\epsilon^{d/2}} \exp\left(-\frac{z^2}{2}\right) g(x-z\epsilon^{1/2}) d^d z \\
&\quad \text{(change of variable: } z \leftarrow t/\epsilon^{1/2}, d^d z \leftarrow d^d t/\epsilon^{d/2} \text{)} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{z^2}{2}\right) \underbrace{\lim_{\epsilon \rightarrow 0} g(x-z\epsilon^{1/2})}_{=g(x)} d^d z \\
&\quad \text{(dominated convergence)} \\
&= g(x) \frac{1}{(2\pi)^{d/2}} \underbrace{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2} z^2\right) d^d z}_{=(2\pi)^{d/2}} \\
&\quad \text{(previous well-known integration result, generalized)} \\
&= g(x) \\
&\Rightarrow \boxed{\mathcal{F} \circ \mathcal{F}^{-1} = \text{id}_{S(\mathbb{R}^d)}}
\end{aligned}$$

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