Commented demonstration of the invertibility of the Fourier transform on Schwartz space. Reworked from:

- F. Schuller QM18 - Fourier Operator;
- Corresponding lectures notes (Simon Rea & Richie Dadhley)

Definition 1. The Fourier operator is the linear map $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ defined for $x \in \mathbb{R}^d$ by:

$$\boxed{(\mathcal{F}(f))(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-ixy)} f(y) d^d y}$$

Result 1. A well-known integration result, recalled in the lecture:

$$\boxed{\int_{\mathbb{R}} \exp{(-\sigma x^2)} dx = \sqrt{\frac{\pi}{\sigma}}}$$

Which generalizes to $\mathbb{R}^d$ to:

$$\boxed{\int_{\mathbb{R}^d} \exp{(-\sigma x^2)} d^d x = \left(\frac{\pi}{\sigma}\right)^{d/2}}$$

Lemma 1. Let $x \in \mathbb{R}^d$ and $z \in \mathbb{C}$, such that $\Re(z) > 0$. Then:

$$\boxed{(\mathcal{F}(x \mapsto \exp{(-\frac{z}{2} x^2)}))(p) = \frac{1}{z^{d/2}} \exp{(-\frac{1}{2z} p^2)}}$$

Theorem 1. $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is invertible and:

$$\boxed{(\mathcal{F}^{-1}(g))(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(ipx)} g(p) d^d p}$$

Proof.

$$
\begin{aligned}
(\mathcal{F}^{-1}(\mathcal{F}(f)))(x) = \;& \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(ipx)} (\mathcal{F}(f))(p) d^d p \\
& \text{(executing the proposed } \mathcal{F}^{-1}\text{)} \\
= \;& \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left(\lim_{\epsilon \to 0} \exp{(-\frac{\epsilon}{2} p^2)} \right)}_{=1} \exp{(ipx)} (\mathcal{F}(f))(p) d^d p \\
& \text{(regulator)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-\frac{\epsilon}{2} p^2)} \exp{(ipx)} (\mathcal{F}(f))(p) d^d p \\
& \text{(Lebesgue dominated convergence)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-\frac{\epsilon}{2} p^2)} \exp{(ipx)} \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-ipy)} f(y) d^d y \right) d^d p \\
& \text{(executing } (\mathcal{F}(f))(p)\text{)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-\frac{\epsilon}{2} p^2)} \underbrace{\exp{(ipx)} \exp{(-ipy)}}_{\exp{(-ip(y-x))}} d^d p \right) f(y) d^d y \\
& \text{(Fubini)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-\frac{\epsilon}{2} p^2)} \exp{(-ipz)} d^d p \right)}_{=:(\mathcal{F}(p \mapsto -\exp{(-\frac{\epsilon}{2} p^2)})(z)} f(z+x) d^d z \\
& \text{(change of variable } z \leftarrow y - x; d^d z \leftarrow d^d y\text{)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left((\mathcal{F}(p \mapsto -\exp{(-\frac{\epsilon}{2} p^2)})(z) \right) f(z+x) d^d z \\
& \text{(Fourier operator definition)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{\epsilon^{d/2}} \exp{(-\frac{1}{2} (\frac{z}{\epsilon^{1/2}})^2)} f(z+x) d^d z \\
& \text{(previous lemma, with } z \leftarrow \epsilon, p \leftarrow z\text{)} \\
= \;& \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\epsilon^{d/2}}{\epsilon^{d/2}} \exp{(-\frac{t^2}{2})} f(t\epsilon^{1/2} + x) d^d t \\
& \text{(change of variable: } t \leftarrow z/\epsilon^{1/2}, d^d t \leftarrow d^d z/\epsilon^{d/2}\text{)} \\
= \;& \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp{(-\frac{t^2}{2})} \underbrace{\lim_{\epsilon \to 0} f(t\epsilon^{1/2} + x)}_{=f(x)} d^d t \\
& \text{(dominated convergence)} \\
= \;& f(x) \frac{1}{(2\pi)^{d/2}} \underbrace{\int_{\mathbb{R}^d} \exp{(-\frac{1}{2} t^2)} d^d t}_{=(2\pi)^{d/2}} \\
& \text{(previous well-known integration result, generalized)} \\
= \;& f(x) \\
\Rightarrow \;& \boxed{\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{id}_{S(\mathbb{R}^d)}}
\end{aligned}
$$

We've proven that the proposed $\mathcal{F}^{-1}$ is the left inverse of $\mathcal{F}$; it remains to prove it is its right inverse. But the process is very similar: we use the same regulator, dominated convergence to swap the limit and the integral, and Fubini to swap the integration order.

Hopefully, without any typos:

$$
\begin{aligned}
(\mathcal{F}(\mathcal{F}^{-1}(g)))(x) = {} & \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ixy)(\mathcal{F}^{-1}(g))(y) d^d y \\
& \text{(executing } \mathcal{F}\text{)} \\
= {} & \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left(\lim_{\epsilon\to 0} \exp(-\frac{\epsilon}{2} y^2)\right)}_{=1} \exp(-ixy)(\mathcal{F}^{-1}(g))(y) d^d y \\
& \text{(regulator)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\frac{\epsilon}{2} y^2) \exp(-ixy)(\mathcal{F}^{-1}(g))(y) d^d y \\
& \text{(Lebesgue dominated convergence)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\frac{\epsilon}{2} y^2) \exp(-ixy) \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(ipy) g(p) d^d p\right) d^d y \\
& \text{(executing the proposed } (\mathcal{F}^{-1}(g))(y)\text{)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\frac{\epsilon}{2} y^2) \underbrace{\exp(ipy)\exp(-ixy)}_{\exp(-iy(x-p))} d^d y\right) g(p) d^d p \\
& \text{(Fubini)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\frac{\epsilon}{2} y^2) \exp(-iyt) d^d y\right)}_{=:(\mathcal{F}(y\mapsto -\exp(-\frac{\epsilon}{2}y^2))(t)} g(x-t) d^d t \\
& \text{(change of variable } t \leftarrow x-p; d^d t \leftarrow d^d p\text{)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathcal{F}(y \mapsto -\exp(-\frac{\epsilon}{2} y^2))(t) g(x-t) d^d t \\
& \text{(Fourier operator definition)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{\epsilon^{d/2}} \exp(-\frac{1}{2}(\frac{t}{\epsilon^{1/2}})^2) g(x-t) d^d t \\
& \text{(previous lemma, with } z \leftarrow \epsilon, p \leftarrow t\text{)} \\
= {} & \lim_{\epsilon\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\epsilon^{d/2}}{\epsilon^{d/2}} \exp(-\frac{z^2}{2}) g(x - z\epsilon^{1/2}) d^d z \\
& \text{(change of variable: } z \leftarrow t/\epsilon^{1/2}, d^d z \leftarrow d^d t/\epsilon^{d/2}\text{)} \\
= {} & \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\frac{z^2}{2}) \underbrace{\lim_{\epsilon\to 0} g(x - z\epsilon^{1/2})}_{=g(x)} d^d z \\
& \text{(dominated convergence)} \\
= {} & g(x) \underbrace{\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\frac{1}{2} z^2) d^d z}_{=(2\pi)^{d/2}} \\
& \text{(previous well-known integration result, generalized)} \\
= {} & g(x) \\
\Rightarrow {} & \boxed{\mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{id}_{S(\mathbb{R}^d)}}
\end{aligned}
$$

□