Commented proof for Parseval's theorem, used in F. Schuller - QM18 - Fourier Operator[1].

Note that this theorem seems to also be referred to as Plancherel identity/theorem, for example in Teschl.

Definition 1. The Fourier operator is the linear map $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ defined for $x \in \mathbb{R}^d$ by:

$$\boxed{(\mathcal{F}(f))(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ixy) f(y) d^d y}$$

Theorem 1. $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is invertible and:

$$\boxed{(\mathcal{F}^{-1}(g))(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(ipx) g(p) d^d p}$$

Proof. See `https://tales.mbivert.com/on-fourier-transform-inverse-proof/`. □

Theorem 2 (Parseval's theorem)**.** Let $f \in S(\mathbb{R}^d)$.

$$\boxed{\int_{\mathbb{R}^d} |\mathcal{F}(f)(p)|^2 d^d p = \int_{\mathbb{R}^d} |f(x)|^2 d^d x}$$

Saying it otherwise, the Fourier operator preserves the L^2 norm.

Proof. We know from $\mathbb{C}$-analysis[2] that, with α^* the $\mathbb{C}$-conjugate of $\alpha \in \mathbb{C}$:

$$\alpha\alpha^* = |\alpha|^2$$

Hence:

$$f(x)(f(x))^* = |f(x)|^2; \qquad \mathcal{F}(f)(p)(\mathcal{F}(f)(p))^* = |\mathcal{F}(f)(p)|^2$$

From the invertibility of the Fourier transform, we have:

$$f(x) = (\mathcal{F}^{-1}(\mathcal{F}(f)))(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(ipx)(\mathcal{F}(f))(p) d^d p$$

And so:

$$(f(x))^* = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ipx)((\mathcal{F}(f))(p))^* d^d p$$

Indeed, we're integrating on $\mathbb{R}^d$, and not on $\mathbb{C}$[3], so for $\phi : \mathbb{R}^d \to \mathbb{C}$, using the linearity of the Lebesgue integral:

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} \phi(p) d^d p\right)^* &:= \left(\int_{\mathbb{R}^d} \Re(\phi(p)) d^d p + i \int_{\mathbb{R}^d} \Im(\phi(p)) d^d p\right)^* \\
&= \int_{\mathbb{R}^d} \Re(\phi(p)) d^d p - i \int_{\mathbb{R}^d} \Im(\phi(p)) d^d p \\
&= \int_{\mathbb{R}^d} \left(\underbrace{\Re(\phi(p)) - i\Im(\phi(p))}_{=:(\phi(p))^*}\right) d^d p \\
&= \int_{\mathbb{R}^d} (\phi(p))^* d^d p
\end{aligned}$$

And this holds in particular for $\phi(p) = \exp(ipx)(\mathcal{F}(f))(p)$. We can then expand the right-hand-side of the theorem:

[1] This is used to prove the boundedness of the Fourier operator on Schwartz space (its operator norm is finite), which allows us to infer via the BLT theorem, as $S(\mathbb{R}^d)$ is dense on $L^2(\mathbb{R}^d)$, that $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is uniquely extensible to $L^2(\mathbb{R}^d)$. The BLT theorem was proved and the operator norm was defined inF. Schuller - QM02 - Banach Spaces

[2] This can be shown by simple calculation, e.g. set $\alpha = x + iy$

[3] The integral of such functions was defined in F. Schuller - QM06 - Integration of measurable functions

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\begin{aligned}
\int_{\mathbb{R}^d} |f(x)|^2 \, d^d x = & \int_{\mathbb{R}^d} f(x)(f(x))^* d^d x \\
& (|f(x)|^2 = f(x)(f(x))^*) \\
= & \int_{\mathbb{R}^d} f(x) \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ipx)((\mathcal{F}(f))(p))^* d^d p \right) d^d x \\
& \text{(inserting our previous expression for } (f(x))^* \text{)} \\
= & \int_{\mathbb{R}^d} ((\mathcal{F}(f))(p))^* \left(\underbrace{\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-ipx) f(x) d^d x}_{=:\mathcal{F}(f)(p)} \right) d^d p \\
& \text{(Fubini)} \\
= & \int_{\mathbb{R}^d} \underbrace{((\mathcal{F}(f))(p))^* \mathcal{F}(f)(p)}_{=|\mathcal{F}(f)(p)|^2} d^d p \\
& \text{(Fourier transform identification)} \\
= & \int_{\mathbb{R}^d} |\mathcal{F}(f)(p)|^2 \, d^d p
\end{aligned}
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$\square$