The Theoretical Minimum

Classical Mechanics - Solutions

I02E01

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M. Bivert

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Exercise 1. *Determine the indefinite integral of each of the following expressions by reversing the process of differentiation and adding a constant.*

$$\begin{aligned} f(t) &= t^4 \\ f(t) &= \cos t \\ f(t) &= t^2 - 2 \end{aligned}$$

Because this is the first integration exercise, we'll go slow. We will "implement" the reversing of the process of differentiation by applying the *fundamental theorem of calculus*, on a few previously established differentiation results; let's recall those:

$$\begin{aligned} \frac{d}{dt}t^n &= nt^{n-1} \\ \frac{d}{dt}\sin t &= \cos t \end{aligned}$$

Let's start by integrating both sides of each equation:

$$\begin{aligned} \int \frac{d}{dt}t^n\,dt &= \int nt^{n-1}\,dt \\ \int \frac{d}{dt}\sin t\,dt &= \int \cos t\,dt \end{aligned}$$

Let's then recall the second form of the *fundamental theorem of calculus* given in the book:

$$\int \frac{d}{dt}f\,dt = f(t) + c, \quad c \in \mathbb{R}$$

So our previous equations can be rewritten as:

$$\begin{aligned} t^n + c &= \int nt^{n-1}\,dt, \qquad c \in \mathbb{R} \\ \sin t + c &= \int \cos t\,dt, \qquad c \in \mathbb{R} \end{aligned}$$

Which are, to syntactical differences, the formulas given in the book. In addition to those, we will also rely on the *linearity of the integration*, which essentially is the combination of the *sum rule for integration* and *multiplication by a constant rule for integration*, both being analogues of what we had for differentiation, and which can be summed up by:

Theorem 1 (linearity of integration)**.**

$$\left(\forall(\alpha,\beta) \in \mathbb{R}^2\right), \left(\forall(\varphi,\psi) \in (C^0)^2\right) \quad \boxed{\int \alpha\varphi + \beta\psi = \alpha\int\varphi + \beta\int\psi}$$

Remark 1. *C^0 refers to the class ("set") of continuous functions; actually, mathematically-wise, it would suffice for the functions to be "partially continuous" so as to be integrable; in the context of physics, requiring them to be continuous is reasonable.*

Remark 2. *We're using the following "shortcut" notation:*

$$\int \varphi = \int \varphi(t)\, dt$$

or for a more involved expression:

$$\int \alpha\varphi + \beta\psi = \int \left(\alpha\varphi(t) + \beta\psi(t)\right)\, dt$$

Proof. We can establish this result, again to syntactical differences, for instance through a similar process as we've just used for t^n and cos, that is, by integrating differentiation results:

$$\begin{array}{rrcl} & \frac{d}{dt}(\alpha\varphi + \beta\psi)(t) & = & \alpha\varphi'(t) + \beta\psi'(t) \\ \Leftrightarrow & \int \frac{d}{dt}(\alpha\varphi + \beta\psi)(t) & = & \int \alpha\varphi'(t) + \beta\psi'(t)dt \\ \Leftrightarrow & (\alpha\varphi + \beta\psi)(t) & = & \int \alpha\varphi'(t) + \beta\psi'(t)dt \\ \Leftrightarrow & \int \alpha\varphi' + \beta\psi' & = & \alpha\int \varphi' + \beta\int \psi' \end{array}$$

$\square$

$f(t) = t^4$
This is a simple application of:

$$\int nt^{n-1}\, dt = t^n + c, \quad c \in \mathbb{R}$$

with $n = 4$; using the *linearity of integration*:

$$\begin{array}{rcl} \int 5t^{5-1}\, dt & = & t^5 + c, \quad c \in \mathbb{R} \\ \Leftrightarrow \int t^4\, dt & = & \boxed{\frac{1}{5}t^5 + c} \end{array}$$

Remark 3. *We can check the result by differentiating it*

$f(t) = \cos t$
An even more direct application of the formulas established earlier:

$$\int \cos t\, dt = \boxed{\sin t + c}, \quad c \in \mathbb{R}$$

Remark 4. *Again, we can check the result using differentiation: we know from earlier that the derivative of a constant is zero, that of sine is cosine, and that the derivative of a sum is the sum of the derivatives.*

$f(t) = t^2 - 2$
Note that there's a special case for:

$$\int nt^{n-1}\, dt = t^n + c, \quad c \in \mathbb{R}$$

when $n = 1$:

$$\int 1 \times t^0\, dt = \int dt = t^1 + c = t + c, \quad c \in \mathbb{R}$$

More generally, by *linearity of the integration*:

$$(\forall \alpha \in \mathbb{R}) \qquad \int \alpha \, dt = \alpha \int dt = \alpha t + c, \quad c \in \mathbb{R}$$

And so we have:

$$\int t^2 - 2 \, dt = \int t^2 \, dt - 2 \int dt = \boxed{\frac{1}{3}t^3 - 2t + c}, \quad c \in \mathbb{R}$$

Which again is elementary to verify by differentiation.