The Theoretical Minimum

Classical Mechanics - Solutions

I02E03

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Treat the expressions from Exercise 1 as expressions for the acceleration of a particle. Integrate them once, with respect to time, and determine the velocities, and a second time to determine the trajectories. Because we will use t as one of the limits of integration we will adopt the dummy integration variable t'. Integrate them from $t' = 0$ to $t' = t$.*

$$\begin{aligned} v(t) &= \int_0^t t'^4 \, dt' \\ v(t) &= \int_0^t \cos t' \, dt' \\ v(t) &= \int_0^t (t'^2 - 2) \, dt' \end{aligned}$$

Conceptually, the exercise seems to be about creating functions from integrals, by having a variable of the integration limit being a function parameter, which forces to use a different name for the integration variable; multiple choices are obviously possible, as long as the result is consistent:

$$v(t) = \int_0^t a(T) \, dT$$

$v(t) = \int_0^t t'^4 dt'$

Let's recall our results from either I02E01 or I02E02. If we work from the former, we first would need to change the variable name from the primitive, say to t', and evaluate the indefinite integral between t and 0. Working from the later, we would need to replace T by t.

$$\begin{aligned} v(t) &= \int_0^t t'^4 dt' \\ &= \left[\frac{1}{5}t'^5 + c\right]_0^t, \quad c \in \mathbb{R} \\ &= \boxed{\frac{1}{5}t^5} \end{aligned}$$

Now we can repeat the same process to compute the position $x(t)$:

$$\begin{aligned} x(t) &= \int_0^t v(t') \, dt' \\ &= \int_0^t \frac{1}{5}t'^5 \, dt' \\ &= \left[\frac{1}{30}t'^6 + c\right]_0^t, \quad c \in \mathbb{R} \\ &= \boxed{\frac{1}{30}t^6} \end{aligned}$$

$v(t) = \int_0^t \cos t' dt'$
Same exact process:

$$\begin{aligned}
v(t) &= \int_0^t \cos t'\, dt' \\
&= [\sin t' + c]_0^t\,, \quad c \in \mathbb{R} \\
&= \boxed{\sin t} \\
x(t) &= \int_0^t v(t')\, dt' \\
&= \int_0^t \sin t'\, dt' \\
&= [-\cos t' + c]_0^t\,, \quad c \in \mathbb{R} \\
&= \boxed{-\cos t}
\end{aligned}$$

$v(t) = \int_0^t (t'^2 - 2)\, dt'$
No surprises here either:

$$\begin{aligned}
v(t) &= \int_0^t (t'^2 - 2)\, dt' \\
&= \left[\frac{1}{3}t'^3 - 2t' + c\right]_0^t, \quad c \in \mathbb{R} \\
&= \boxed{\frac{1}{3}t^3 - 2t} \\
x(t) &= \int_0^t v(t')\, dt' \\
&= \int_0^t \frac{1}{3}t'^3 - 2t'\, dt' \\
&= \left[\frac{1}{12}t'^4 - t'^2 + c\right]_0^t, \quad c \in \mathbb{R} \\
&= \boxed{\left(\frac{1}{12}t^2 - 1\right)t^2}
\end{aligned}$$