The Theoretical Minimum

Classical Mechanics - Solutions

L02E01

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Calculate the derivatives of each of these functions.*

$$\begin{aligned} f(t) &= t^4 + 3t^3 - 12t^2 + t - 6 \\ g(x) &= \sin x - \cos x \\ \theta(\alpha) &= e^\alpha + \alpha \ln \alpha \\ x(t) &= \sin^2 t - \cos t \end{aligned}$$

Remark 1. *Those are exactly the functions graphed in I01E01*

$f(t) = t^4 + 3t^3 - 12t^2 + t - 6$

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To find the derivative of f, we need to apply four rules that were mentioned (the first one was proved) in the book:

1. the formula for the derivative of a general power; for $n \in \mathbb{N}$:
$$\frac{d}{dt}(t^n) = nt^{n-1}$$
2. the fact that the derivative of a constant is zero; for $c \in \mathbb{R}$:
$$\frac{d}{dt}c = 0$$
3. the fact that the derivative of a constant times a function is the same as the constant times the derivative of the function; for $c \in \mathbb{R}$ and φ a function of t:
$$\frac{d}{dt}(c\varphi) = c\frac{d}{dt}\varphi$$
4. the *sum rule* (also referred to as the *linearity of differentiation* in mathematics); for both φ and ψ functions of t:
$$\frac{d}{dt}(\varphi + \psi) = \frac{d}{dt}\varphi + \frac{d}{dt}\psi$$

Remark 2. *Two simpler notations are common to denote the derivative of a function of a single variable:*

$$\frac{d}{dt}\varphi = \varphi' = \dot{\varphi}$$

While the first one is used mostly in mathematics, for abstract functions, the second one is used almost exclusively in physics, to denote a time derivative. We'll use them both in such ways from now on. For instance, as this exercise is rather mathematical, we'll use the prime notation.

While we're here, bear in mind that the prime notation can also be used to denote the derivative of a more or less complex expression, wrapped in parenthesis, e.g.:

$$(\varphi(x) + \psi(x) + \cos^2 x)' = \frac{d}{dx}(\varphi(x) + \psi(x) + \cos^2 x)$$

By application of the *sum rule*, the rule regarding the derivative of a constant, and the rule regarding a function multiplied by a constant:

$$f'(t) = (t^4)' + 3(t^3)' - 12(t^2)' + (t)' - 0$$

Then, it's just a matter of applying the formula for the derivative of a general power to each individual term:

$$\boxed{f'(t) = 4t^3 + 9t^2 - 24t + 1}$$

$g(x) = \sin x - \cos x$

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In addition the the previously mentioned *sum rule*, we will also need two more rules to compute the derivative of g, also presented in the book, regarding the derivative of sin and cos:

$$\cos'(t) \quad = -\sin(t); \qquad \sin'(t) \quad = \cos(t)$$

We then have successively:

$$\begin{aligned} g'(x) &= \sin'(x) - \cos'(x) \\ &= \boxed{\cos(x) + \sin(x)} \end{aligned}$$

$\theta(\alpha) = e^\alpha + \alpha \ln \alpha$

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To compute θ', in addition to the *sum rule* and the formula for a general power (with $n = 1$), we need two additional rules pertaining to the derivatives of both the exponential and the log:

$$(e^t)' \quad = \quad e^t; \qquad \ln'(t) \quad = \quad \frac{1}{t}$$

And the *product rule*; for φ and ψ functions of t:

$$\frac{d}{dt}(\varphi\psi) = \varphi'\psi + \varphi\psi'$$

Where all those rules were mentioned in the book. We then obtain:

$$\begin{aligned} \theta'(\alpha) &= (e^\alpha)' + (\alpha \ln(\alpha))' \\ &= e^\alpha + (\frac{d}{d\alpha}\alpha)\ln(\alpha) + \alpha \ln'(\alpha) \\ &= \boxed{e^\alpha + \ln(\alpha) + 1} \end{aligned}$$

$x(t) = (\sin t)^2 - \cos t$

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Besides the *sum rule*, the rules regarding the derivative of cos and sin, and the formula for the derivative of a general power, we will only need a single new rule to compute x', the *chain rule*; for φ a function of t, and ψ and function whose domain (input) is the codomain (output) of φ:

$$\frac{d}{dt}(\psi \circ \varphi) = \frac{d}{dt}(\psi(\varphi(t)) = \varphi'(t)\psi'(\varphi(t))$$

Then,

$$\begin{aligned} x'(t) &= ((\sin t)^2)' - (\cos t)' \\ &= (\sin t)'(u \mapsto u^2)'(\sin t) + \sin t \\ &= \cos t(u \mapsto 2u)(\sin t) + \sin t \\ &= 2\cos t \sin t + \sin t \\ &= \boxed{(1 + 2\cos t)\sin t} \end{aligned}$$

Remark 3. *We used a bit of mathematical notation to avoid us the need to explicitly name the function which squares its argument. More explicitly, to compute the derivative of $\mu(t) = \sin^2 t$, we could have define $\nu(u) = u^2$. Then, $\mu(t) = \nu(\sin(t))$, and $\nu'(u) = 2u$; by the chain rule*

$$(\sin^2 t)' = \mu'(t) = (\sin t)'\nu'(\sin(t)) = 2\cos t \sin t$$

Remark 4. *Instead of using the chain rule to compute the derivative of* $\sin^2 t$*, we could instead have used the product rule:*

$$(\sin^2 t)' = (\sin t \times \sin t)' = (\sin t)' \sin t + \sin t(\sin t)' = 2 \sin t(\sin t)' = 2 \sin t \cos t$$