The Theoretical Minimum

Classical Mechanics - Solutions

L02E03

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Use the chain rule to find the derivatives on each of the following functions:*

$$\begin{aligned} g_0(t) &= \sin(t^2) - \cos(t^2) \\ \theta_0(\alpha) &= e^{3\alpha} + 3\alpha \ln(3\alpha) \\ x_0(t) &= \sin^2(t^2) - \cos(t^2) \end{aligned}$$

Remark 1. *We've slightly altered the functions names compared to what they have in the book. The reason will be apparent very soon.*

Remark 2. *As in previous exercises (e.g. L02E01), we'll be using some additional mathematical notation to avoid us the need to define functions. The meaning should be obvious from the context.*

We will also use the prime notation to denote differentiation, as we also did in earlier exercises.

There are two main ways of proceeding: either applying the *chain rule* to each individual term of each function, for instance to compute g', we would fist compute $(\sin(t^2))' = (\sin((u \mapsto u^2)(t)))'$, using the chain rule, then $(\cos(t^2))'$ in a similar fashion, etc.

Or, and we're going to use this approach as it's likely to have been expected one, we can consider each function "globally" as a composition of two smaller functions. While doing so, we will find that g_0, θ_0 and x_0 are respectively created by composing functions f, θ and x of an earlier exercise, with two simple functions $(v \mapsto v^2)$ and $(v \mapsto 3v)$:

$$\begin{aligned} g_0(t) &= (\underbrace{(u \mapsto \sin(u) - \cos(u))}_{g} \circ (v \mapsto v^2))(t) \\ \theta_0(\alpha) &= (\underbrace{(u \mapsto e^u + u\ln(u))}_{\theta} \circ (v \mapsto 3v))(t) \\ x_0(t) &= (\underbrace{(u \mapsto \sin^2(u) - \cos(u))}_{x} \circ (v \mapsto v^2))(t) \end{aligned}$$

Let's remember the derivative of g, θ and x, that we've computed in L02E01:

$$\begin{aligned} g'(x) &= \cos(x) + \sin(x) \\ \theta'(\alpha) &= e^\alpha + \ln(\alpha) + 1 \\ x'(t) &= (1 + 2\cos t)\sin t \end{aligned}$$

Finally, let's recall the *chain rule*:

$$\frac{d}{dt}(\psi \circ \varphi) = \frac{d}{dt}(\psi(\varphi(t)) = \varphi'(t)\psi'(\varphi(t))$$

Then, the derivative of $(v \mapsto v^2)$ and $(v \mapsto 3v)$ being respectively $(v \mapsto 2v)$ and $(v \mapsto 3)$ (constant function), we have the following derivatives for our functions:

$$
\begin{aligned}
g_0'(t) &= \boxed{2t(\cos t^2 + \sin t^2)} \\
\theta_0'(\alpha) &= \boxed{3(e^{3\alpha} + \ln(3\alpha) + 1)} \\
x_0'(t) &= \boxed{2t((1 + 2\cos t^2)\sin t^2)}
\end{aligned}
$$