The Theoretical Minimum
Classical Mechanics - Solutions
L02E04

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Prove the sum rule (fairly easy), the product rule (easy if you know the trick), and the chain rule (fairly easy).*

You may want to refer to a *real* mathematical textbook for a finer, more rigorous treatment of those exercises. We'll give reasonably solid proofs, that should be sufficient in the context of an introductory physics textbook. A interesting middle-ground, that we won't explore here, would be to study the proofs that one can obtain from an alternative (but equivalent) formulation of the derivative known as the Carathéodory's derivative, which allows for simple proofs of such results.

Let's start by recalling how differentiation is defined.

Definition 1. *A function $\varphi : E \to \mathbb{R}$ is said to be differentiable at a point $e \in E$ if the following limit exists:*

$$\varphi'(e) = \frac{d}{dx}\varphi(e) = \boxed{\lim_{\epsilon\to 0} \frac{\varphi(e+\epsilon) - \varphi(e)}{\epsilon}}$$

If this limit exists for all points x of E (we note, $(\forall x \in E)$), then φ is said to be differentiable on E, or simply differentiable. The function which associate to each points of E this limit is called the derivative of φ, and is called φ.

Theorem 1 (sum rule)**.** *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi + \psi)' = \varphi' + \psi'}$$

Proof. We have, by definition of the differentiation, and after re-ordering the terms

$$\begin{aligned}
(\forall x \in E), \quad (\varphi+\psi)'(x) &= \lim_{\epsilon\to 0} \frac{(\varphi+\psi)(x+\epsilon) - (\varphi+\psi)(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon) + \psi(x+\epsilon) - \varphi(x) - \psi(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \left(\frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} + \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} \right) \\
&= \lim_{\epsilon\to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} + \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} \\
&= \boxed{\varphi'(x) + \psi'(x)}
\end{aligned}$$

□

Remark 1. *As a rigorous proof is a bit tedious*[1]*, we* *<u>assumed</u>* *for the last step that a limit of a sum is the sum of the limits,* *<u>when all the involved limits exist</u>* *(which is the case here, because those limits are equivalent to saying our functions are differentiable, which they are, per hypothesis)*

$$\lim_{x\to a} (\varphi(x) + \psi(x)) = \lim_{x\to a} \varphi(x) + \lim_{x\to a} \psi(x))$$

[1] if you want one, have a look at *Paul's Online Notes*: `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`

Theorem 2 (product rule)**.** *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi\psi)' = \varphi'\psi + \varphi\psi'}$$

Proof. This is a simple and often used theorem, but unfortunately, the proof of it is a bit "magic": if we start by applying the definition of the differentiation to $(\varphi\psi)'$, we have to introduce a well-crafted term (in the form $-a + a = 0$) so as to factorize things to meet our goal. We would furthermore be implicitly assuming that $(\varphi\psi)'$ exists, but we have no guarantee of it.

We can solve those issues by starting from the definition of the differentiation of $\varphi'\psi + \varphi\psi'$, but assuming we already know the result we're trying to prove, is conceptually clumsy. Perhaps calling this a "verification" rather than a proof would be more correct then.

Using the former derivation, we have:

$$\begin{aligned}
(\forall x \in E), \quad (\varphi\psi)'(x) &= \lim_{\epsilon \to 0} \frac{(\varphi\psi)(x+\epsilon) - (\varphi\psi)(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x+\epsilon)\psi(x+\epsilon) - \varphi(x)\psi(x)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x+\epsilon)\psi(x+\epsilon) - \varphi(x)\psi(x) \overbrace{-\varphi(x+\epsilon)\psi(x) + \varphi(x+\epsilon)\psi(x)}^{=0}}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\varphi(x+\epsilon)(\psi(x+\epsilon) - \psi(x)) + \psi(x)(\varphi(x+\epsilon) - \varphi(x))}{\epsilon} \\
&= \lim_{\epsilon \to 0} \varphi(x+\epsilon)\frac{(\psi(x+\epsilon) - \psi(x))}{\epsilon} + \psi(x) \lim_{\epsilon \to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} \\
&= \left(\lim_{\epsilon \to 0} \varphi(x+\epsilon)\right) \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} + \psi(x) \lim_{\epsilon \to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} \\
&= \boxed{\varphi(x)\psi'(x) + \psi(x)\varphi'(x)}
\end{aligned}$$

$\square$

Remark 2. *We <u>assumed</u> another result on limits* [2]*, again assuming individual limits exists (and they do in our case, as our functions are differentiable, and (thus) continuous):*

$$\lim_{x \to a} (\varphi(x)\psi(x)) = \lim_{x \to a} \varphi(x) \times \lim_{x \to a} \psi(x))$$

Theorem 3 (chain rule)**.** *Let $\varphi, \psi : E \to \mathbb{R}$, both differentiable on E. Then,*

$$\boxed{(\varphi \circ \psi)' = \psi' \times (\varphi' \circ \psi)}$$

Proof. Again, the proof is a bit "magical" in that we're going to multiply by a well-crafted term, of the form $a/a = 1$:

$$\begin{aligned}
(\forall x \in E), \, (\varphi \circ \psi)(x)) &= \lim_{\epsilon \to 0} \frac{(\varphi \circ \psi)(x+\epsilon) - (\varphi \circ \psi)(x))}{\epsilon} \\
&= \lim_{\epsilon \to 0} \left(\frac{\varphi(\psi(x+\epsilon)) - \varphi(\psi(x)))}{\epsilon} \times \overbrace{\frac{\psi(x+\epsilon) - \psi(x)}{\psi(x+\epsilon) - \psi(x)}}^{=1} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{\varphi(\psi(x+\epsilon)) - \varphi(\psi(x)))}{\psi(x+\epsilon) - \psi(x)} \times \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \frac{\varphi(\psi(x+\epsilon)) - \varphi(\psi(x)))}{\psi(x+\epsilon) - \psi(x)} \times \underbrace{\lim_{\epsilon \to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon}}_{\psi'(x)}
\end{aligned}$$

[2] If you want a rigorous proof of it, you can refer to the same resource as before: `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`

Again for that last step, we've used the aforementioned rule on products of existing limits. To conclude, we need to compute the first limit: let's define $h = \psi(x+\epsilon) - \psi(x) \Leftrightarrow \psi(x+\epsilon) = \psi(x) + h$. Note that $\epsilon \to 0 \Rightarrow h \to 0$. So the first limit can be rewritten:

$$\lim_{h \to 0} \frac{\varphi(\psi(x) + h) - \varphi(\psi(x)))}{h} \triangleq \varphi'(\psi(x))$$

$\square$

Now a problem with this previous proof is that it is invalid if ψ is a constant function for example, because $\psi(x+\epsilon) - \psi(x)$ is zero, and we're dividing by zero when performing our magical "multiplication" by 1. If you're interested, you can find an alternative proof using the other, equivalent form of the derivative here: `https://www.youtube.com/watch?v=COLwYhEAt7Q`.