The Theoretical Minimum

Classical Mechanics - Solutions

L02E05

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Remark 1. *This is a WIP; some intermediate results are missing.*

Exercise 1. *Prove each of the formulas in Eq.s* (2)*. Hint: Look up trigonometric identities and limit properties in a reference book.*

Let's recall the formulas of Eq.s (2):

$$\begin{aligned}\frac{d}{dt}(\sin t) &= \cos t\\ \frac{d}{dt}(\cos t) &= \sin t\\ \frac{d}{dt}(e^t) &= e^t\\ \frac{d}{dt}(\ln t) &= \frac{1}{t}\end{aligned}$$

Remark 2. *Interestingly, there are multiple ways of defining those functions*[1]*. As a result, there are different ways to compute the derivatives, depending on which definitions we choose.*

As the definitions given in the book for sin, cos *and the exponential are rather standard, we'll recall them and use those. The natural logarithm hasn't been clearly defined though, so we'll have to do it.*

Remark 3. *The book suggests to look up trigonometric identities and common properties in some reference material, but we're going to take the time to prove almost all intermediate results here.*

As far as I can tell, to compute the derivative of either sin *or* cos*, starting from a geometrical (triangle-based) definition, while keeping the definition of π as a measure of angle in radians, one need to start from basic Euclidean geometry, derive some specific limits, and lightly touch on elementary integration results. This is a good occasion to refresh basic real analysis.*

To say it otherwise, we will go ***far beyond*** *what is expected for this exercise by the authors. That being said, take what follows with a grain of salt: there's a lot of results, which can be subtle, so you may want to refer to a more thorough treatment by real mathematicians in case of doubts.*

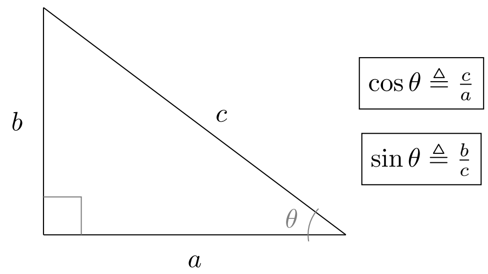
Let's start by recalling that a function $\varphi : E \to \mathbb{R}$ is said to be differentiable at a point $e \in E$ if the following limit exists:

$$\varphi'(e) = \frac{d}{dx}\varphi(e) = \boxed{\lim_{\epsilon \to 0} \frac{\varphi(e+\epsilon) - \varphi(e)}{\epsilon}}$$

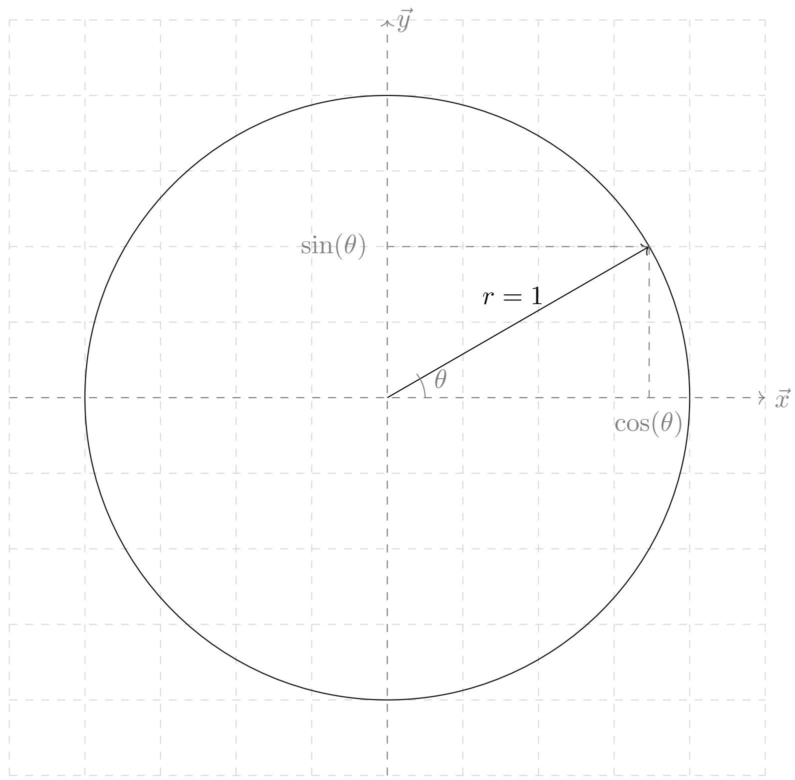
$d \sin t / dt$

Then, let's remind ourselves of the common definitions of cos and sin, in the context of a right triangle:

[1]For instance, consider this page containing 6 equivalent definitions of the exponential: `https://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function`



In particular, we can identify points on the unit-circle by the angle between the x axis and the radius connecting the center of the circle to such points. Then, each point will then be located in the xy-plane as $(\cos\theta, \sin\theta)$, where θ is the angle previously described, associated to the point.



Note that we have:

Theorem 1.

$$(\forall x \in \mathbb{R}), \quad \boxed{\sin^2 x + \cos^2 x = 1}$$

Proof. This follows immediately from the Pythagorean theorem applied to the right triangle formed by $r = 1$, $\cos\theta$ and $\sin\theta$. □

We'll need this later: this is but a variant of the previous result where the circle isn't restricted to being unitary:

Theorem 2 (equation of a circle)**.** *The points $(x, y) \in \mathbb{R}^2$ describing a circle of radius r centered at the origin O are tied by the following equation*

$$\boxed{x^2 + y^2 = r^2}$$

Proof. This follows directly from Pythagorean's theorem □

Remark 4. *In particular, as r is a constant, this mean we can express y as a function of x:*

$$\boxed{y(x) = \sqrt{r^2 - x^2}}; \ x \in [-r, r]$$

In order to establish $\sin'$, we will need a few intermediate results that we're going to prove now. First will be to find a formula for $\sin(\alpha+\beta)$. Indeed, if you try to apply the definition of the derivative to sin, you should see a $\sin(x+\epsilon)$: we will need to have it expressed differently to develop the proof.

$$\sin'(x) \triangleq \lim_{\epsilon\to 0} \frac{\sin(x+\epsilon)-\sin x}{\epsilon}$$

Theorem 3 $(\sin(\alpha+\beta))$**.**

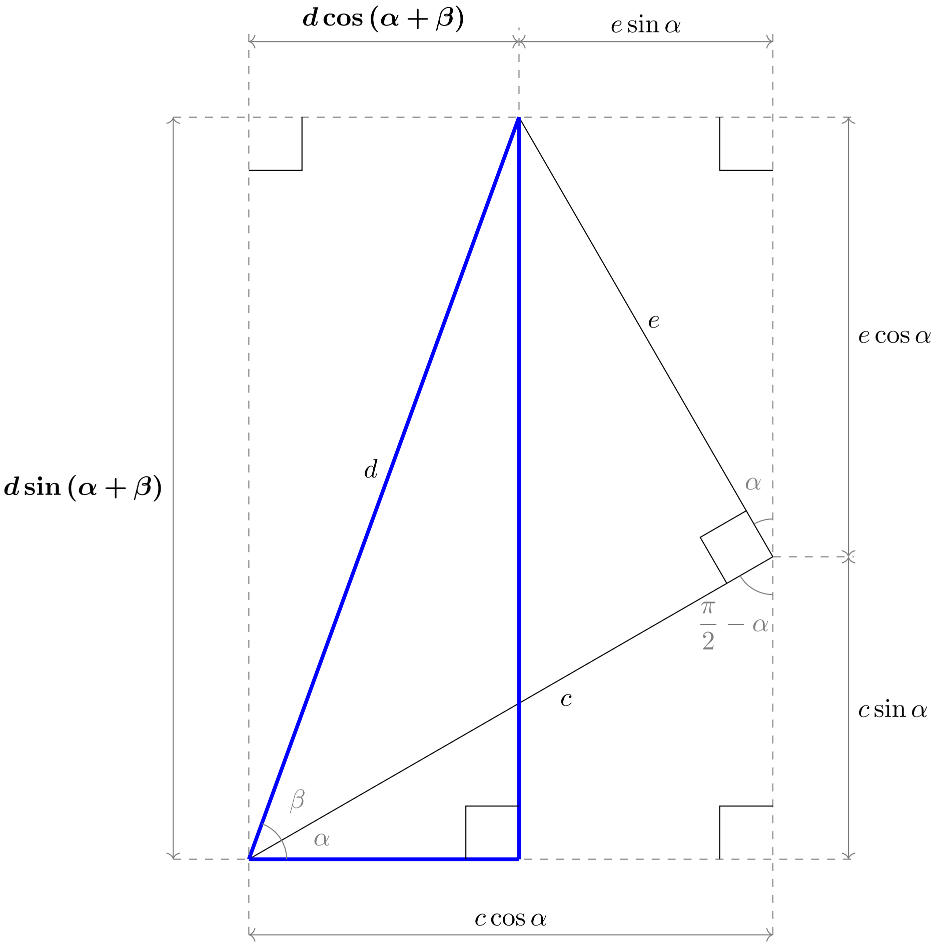
$$(\forall(\alpha,\beta)\in\mathbb{R}^2), \quad \boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta} \tag{1}$$

There's a also a formula for the cosine of a sum of angles, that we will need later, for the derivative of cos, but that will be rather immediate to prove along the one regarding the sine of a sum of angles.

Theorem 4 $(\cos(\alpha+\beta))$**.**

$$(\forall(\alpha,\beta)\in\mathbb{R}^2), \quad \boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta} \tag{2}$$

Proof. This will be a "visual proof". Besides the aforementioned definitions of sin and cos, we will also use the "fact" that sum of angles in a right triangle is π, which is actually the *triangle postulate*, an axiom of Euclidean geometry, equivalent to the parallel postulate.



In the previous picture, considering the right triangle formed by c, d and e, we have:

$$c = d\cos\beta; \qquad e = d\sin\beta$$

If we look at the blue/thick triangle right triangle (of hypotenuse d, with an angle of $\beta+\alpha$, and whose other sides are created by projecting the point formed by d and e down to the bottom), we find a new relation, to which we can inject our previous results for c and e:

$$\begin{aligned} d\sin(\alpha+\beta) &= c\sin\alpha + e\cos\alpha \\ \Leftrightarrow \qquad &= (d\cos\beta)\sin\alpha + (d\sin\beta)\cos\alpha \\ \Leftrightarrow \qquad \sin(\alpha+\beta) &= \boxed{\cos\beta\sin\alpha + \sin\beta\cos\alpha} \end{aligned}$$

In the same blue/thick triangle, we can also establish a relation for $\cos(\alpha+\beta)$, using the same definition of c and e as before to conclude:

$$\begin{aligned} d\cos(\alpha+\beta) &= c\cos\alpha - e\sin\alpha \\ \Leftrightarrow \qquad &= (d\cos\beta)\cos\alpha - (d\sin\beta)\sin\alpha \\ \Leftrightarrow \qquad \cos(\alpha+\beta) &= \boxed{\cos\beta\cos\alpha - \sin\beta\sin\alpha} \end{aligned}$$

□

Here's an immediate consequence that we'll need in the future.

Theorem 5 (trigonometric shifts)**.** *Let $x \in \mathbb{R}$.*

$$\boxed{\sin(x+\frac{\pi}{2}) = \cos x; \qquad \cos(x+\frac{\pi}{2}) = -\sin x}$$

Proof. If we apply our previous formulas (1) and (2) regarding respectively the sine and cosine of a sum of two angles, in the case where one angle is $\pi/2$, we have:

$$\begin{aligned} (\forall x\in\mathbb{R}),\quad \sin(x+\frac{\pi}{2}) &= \sin x \underbrace{\cos\frac{\pi}{2}}_{=0} + \cos x \underbrace{\sin\frac{\pi}{2}}_{=1} \\ &= \boxed{\cos x} \\ (\forall x\in\mathbb{R}),\quad \cos(x+\frac{\pi}{2}) &= \cos x \underbrace{\cos\frac{\pi}{2}}_{=0} - \sin x \underbrace{\sin\frac{\pi}{2}}_{=1} \\ &= \boxed{-\sin x} \end{aligned}$$

□

Remark 5. *We could derive more similar formulas, but those are the only ones we'll need.*

Now if you try to write down $\sin'$ as previously suggested, and if you decompose $\cos(x+\epsilon)$ with the formula (1), you see that yields two limits:

$$\begin{aligned} \sin'(x) &\triangleq \lim_{\epsilon\to 0} \frac{\sin(x+\epsilon)-\sin x}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\sin x\cos\epsilon + \cos x\sin\epsilon - \sin x}{\epsilon} \\ &= \lim_{\epsilon\to 0} \left(\frac{\sin x\cos\epsilon - 1}{\epsilon} + \frac{\cos x\sin\epsilon}{\epsilon}\right) \end{aligned}$$

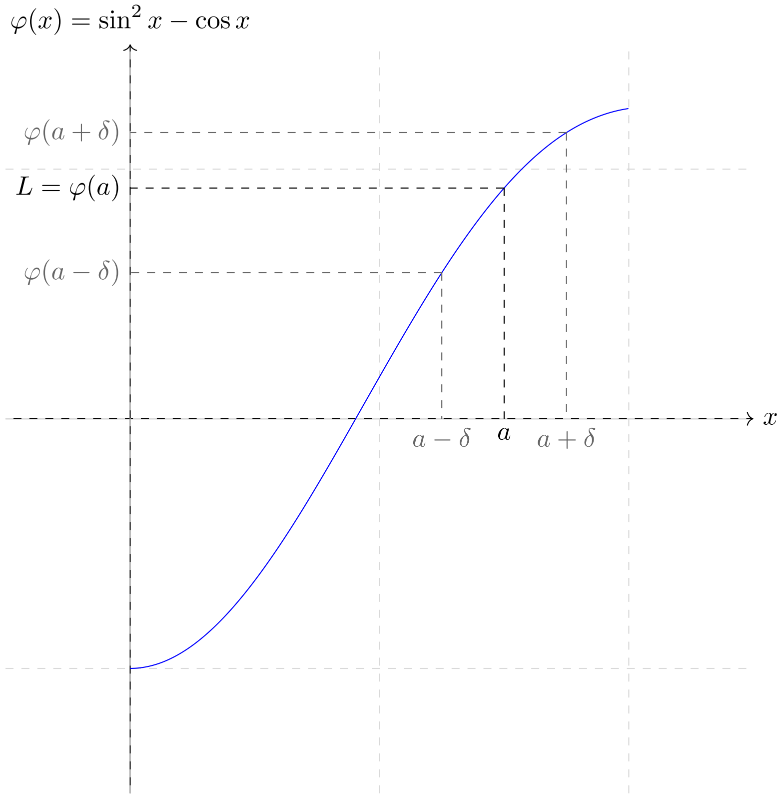
As we've already explained in L02E04, we can recursively split the previous limits, assuming each individual limit exists. Let's digress and review how to prove such results on limits. Starting with the definition of a limit:

Definition 1 ((ϵ,δ)-definition of a limit)**.** *Let $\varphi:\mathbb{R}\to\mathbb{R}$. Saying that $\lim_{x\to a}\varphi(x) = L$ is the same as saying:*

$$(\forall\epsilon>0),\ (\exists\delta\in\mathbb{R}),\ (\forall x\in\mathbb{R})\ (|x-a|<\delta) \Rightarrow (|\varphi(x)-L|<\epsilon)$$

The definition is dense, so let's unpack it. Let's fix ϵ to some value close to zero, say 0.1, or 0.0001.

Now the idea is that, for this ϵ, we will always be able to find a distance δ such that if we pick an x between $a-\delta$ and $a+\delta$ (i.e. $|x-a|<\delta$), then $\varphi(x)$ will be between $L-\epsilon$ and $L+\epsilon$ (i.e. $|\varphi(x)-L|<\epsilon$).



But, this is true for all strictly positive ϵ. So in particular, this is true for an ever so smaller ϵ. In other words, regardless of how close we want $\varphi(x)$ and L to be, we will always be able to achieve it if we bring x and a close enough.

Alright, there's just one more thing we need before proving the sum rule, and that's the triangle inequality. This inequality is rooted in euclidean geometry: it states that the sum of the length of any two sides of a triangle is greater or equal than the length of the remaining side.

Theorem 6 (triangle inequality)**.**

$$\boxed{(\forall (x,y)\in\mathbb{R}^2),\quad |x+y|\leq |x|+|y|}$$

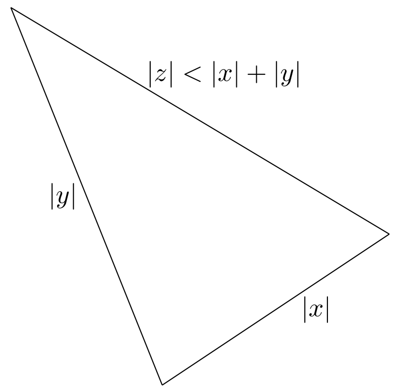


Figure 1: Unless the triangle is degenerate (i.e. all its edges are colinear), the length of the longest side of a triangle is strictly smaller than the length of the two other sides. It's trivially true for the shorter sides. If we allow triangles to be degenerates, then it's true for any sides of any triangle.

Proof. Let's start by recalling the definition of the absolute value:

$$\boxed{x \in \mathbb{R}; \quad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}}$$

It follows that:

$$|x+y| = \begin{cases} x+y & \text{if } (x+y) \geq 0 \\ -(x+y) = -x-y & \text{otherwise} \end{cases}$$

And that, for any $(x, y) \in \mathbb{R}^2$

$$x <= |x|; \quad -x \leq |x|; \quad y <= |y|; \quad -y \leq |y|;$$

So:

$$\left(x + y \leq |x| + |y|; \quad -(x+y) = -x - y \leq |x| + |y|\right) \Leftrightarrow \boxed{|x+y| \leq |x| + |y|}$$

□

Let's jump into the sum rule:

Theorem 7 (sum rule for limits)**.** *Assuming the two following limits exists:*

$$\lim_{x \to a} \varphi(x); \qquad \lim_{x \to a} \psi(x)$$

Then:

$$\boxed{\lim_{x \to a} (\varphi(x) + \psi(x)) = \lim_{x \to a} \varphi(x) + \lim_{x \to a} \psi(x)}$$

Proof. Most limits proofs are presented in a "confusing" way, starting with unexplained values that ends up doing exactly what we want. That happens when mathematicians have thought and drafted the proof in reverse order, but present it in the "right" order. We're going to use the "wrong" order here, for clarity; it should be immediate to check that there's no logical issues anyway. Just bear in mind that "reversed implications" are rather unorthodox.

Let's start by defining a few things:

$$\lim_{x \to a} \varphi(x) = L_1; \qquad \lim_{x \to a} \psi(x) = L_2$$

$$\lim_{x \to a} (\varphi(x) + \psi(x)) = L$$

Let's explicit the two first limits via the $(\epsilon - \delta)$-definition:

$$(\forall \epsilon_1 > 0), \, (\exists \delta_1 \in \mathbb{R}), \, (\forall x \in \mathbb{R}) \, (|x - a| < \delta_1) \Rightarrow (|\varphi(x) - L_1| < \epsilon_1)$$

$$(\forall \epsilon_2 > 0), (\exists \delta_2 \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta_2) \Rightarrow (|\psi(x) - L_2| < \epsilon_2)$$

Essentially, what we want to prove is then $L = L_1 + L_2$. If this is true, this means means that for an $\epsilon \in \mathbb{R}$, we should be able to find a δ such that:

$$\begin{aligned}
& |\varphi(x) + \psi(x) - L| && < \epsilon && \\
\Leftrightarrow\ & |\varphi(x) + \psi(x) - (L_1 + L_2)| && < \epsilon && \text{(assumption)} \\
\Leftrightarrow\ & |(\varphi(x) - L_1) + (\psi(x) - L_2)| && < \epsilon && \\
\Leftarrow\ & |\varphi(x) - L_1| + |\psi(x) - L_2| && < \epsilon && \text{(triangular inequality)} \\
\Leftrightarrow\ & |\varphi(x) - L_1| + |\psi(x) - L_2| && < \epsilon_1 + \epsilon_2 && (\epsilon = \epsilon_1 + \epsilon_2)
\end{aligned}$$

$$\Leftarrow \begin{cases} |\varphi(x) - L_1| < \epsilon_1 = \epsilon/2 \\ |\psi(x) - L_2| < \epsilon_2 = \epsilon/2 \end{cases} \qquad \text{(all numbers are positive)}$$

$$\Leftarrow \Big(|x - a| < \delta = \min(\delta_1, \delta_2)\Big)$$

To make things very clear, if we choose such a δ, then:

$$|x - a| < \delta \Leftrightarrow \begin{cases} |x - a| < \delta_1 \\ |x - a| < \delta_2 \end{cases}$$

Which means both limits will hold. Furthermore, given any real number, say ϵ_1, we can always choose to represent it as $\epsilon/2$, for $\epsilon \in \mathbb{R}$ too.

And this concludes the proof: if you look at the beginning and end of the previous derivation, we've found:

$$(\forall \epsilon > 0), (\exists \delta \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta) \Rightarrow \left(|\varphi(x) + \psi(x) - \underbrace{L_1 + L_2}_{L}| < \epsilon \right)$$

Which by definition means:

$$\lim_{x \to a} (\varphi(x) + \psi(x)) = L_1 + L_2$$

□

The following is not necessary here, but it's more involved, so good practice:

Theorem 8 (product rule for limits)**.** *Assuming the two following limits exists:*

$$\lim_{x \to a} \varphi(x); \qquad \lim_{x \to a} \psi(x)$$

Then:

$$\boxed{\lim_{x \to a} (\varphi(x)\psi(x)) = \lim_{x \to a} \varphi(x) \lim_{x \to a} \psi(x)}$$

Proof. We'll use the same unorthodox presentation; again, let's start by defining a few things:

$$\lim_{x \to a} \varphi(x) = L_1; \qquad \lim_{x \to a} \psi(x) = L_2$$

$$\lim_{x \to a} (\varphi(x)\psi(x)) = L$$

What we want to prove this time is that $L = L_1 L_2$. If this is true, this means means that for an $\epsilon \in \mathbb{R}$, we should be able to find a δ such that:

$$\begin{aligned}
& |\varphi(x)\psi(x) - L| && < \epsilon \\
\Leftrightarrow\ & |\varphi(x)\psi(x) - L_1 L_2| && < \epsilon
\end{aligned}$$

Well, this is embarrassing: we can't really follow through as we did before. Can we find an expression that we could algebraically connect to $\varphi(x)\psi(x) - L_1 L_2$?

Let's make a guess, and try the following product $(\varphi(x) - L_1)(\psi(x) - L_2)$. When developed, clearly it contains the previous expression that we desperately try to find a path to, plus two other terms:

$$\varphi(x)\psi(x) - L_1\psi(x) - L_2\varphi(x) + L_1L_2$$

And we can see that in the context of x being arbitrarily close to a, we have, by the sum rule:

$$L_1\psi(x) \to L_1L_2; \qquad L_2\varphi(x) \to L_2L_1$$

Which means, when $x \to a$, this product should be equal to $\varphi(x)\psi(x) - L_1L_2$. In summary, if we can prove that $(\varphi(x) - L_1)(\psi(x) - L_2)$ can be made as small as we want as $x \to a$, we should have a proof. Let's, then study the corresponding limit (that we expect to be zero):

$$\lim_{x\to a}(\varphi(x) - L_1)(\psi(x) - L_2)$$

First, note that we have:

$$\begin{array}{rcccl} \lim\limits_{x\to a}(\varphi(x) - L_1) & = & (\lim\limits_{x\to a}\varphi(x)) - L_1 & = & 0 \\ \lim\limits_{x\to a}(\psi(x) - L_2) & = & (\lim\limits_{x\to a}\psi(x)) - L_2 & = & 0 \end{array}$$

Which means, translated in the $(\epsilon - \delta)$ formalism (as before, we choose the same ϵ so as to control both limits at once)

$$(\forall \epsilon > 0),\ (\exists \delta_1 \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x - a| < \delta_1) \Rightarrow (|\varphi(x) - L_1 - 0| < \epsilon)$$
$$(\forall \epsilon > 0),\ (\exists \delta_2 \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x - a| < \delta_2) \Rightarrow (|\psi(x) - L_2 - 0| < \epsilon)$$

Let $\epsilon > 0$; let's choose $\delta = \min(\delta_1, \delta_2)$. Then

$$|x - a| < \delta \Leftrightarrow \begin{cases} |x - a| < \delta_1 \\ |x - a| < \delta_2 \end{cases}$$

Which means we have both:

$$\begin{cases} |\varphi(x) - L_1| < \epsilon \\ |\psi(x) - L_2| < \epsilon \end{cases}$$

All involved numbers are positive, so let's multiply both lines:

$$|\varphi(x) - L_1| \times |\psi(x) - L_2| < \epsilon^2$$
$$\Leftrightarrow |(\varphi(x) - L_1)(\psi(x) - L_2)| < \epsilon^2$$

But if ϵ can be chosen arbitrarily in $\mathbb{R}_+^*$, then so can $\epsilon' = \epsilon^2$. So we've just proved that

$$(\forall \epsilon' > 0),\ (\exists \delta \in \mathbb{R}),\ (\forall x \in \mathbb{R})\ (|x - a| < \delta) \Rightarrow (|(\varphi(x) - L_1)(\psi(x) - L_2) - 0| < \epsilon')$$

That is:

$$\lim_{x\to a}(\varphi(x) - L_1)(\psi(x) - L_2) = 0$$

From what we've already said, this concludes the proof, but let's make it clear:

$$\begin{array}{rrcl} & \varphi(x)\psi(x) - L_1\psi(x) - L_2\varphi(x) + L_1L_2 & = & (\varphi(x) - L_1)(\psi(x) - L_2) \\ \Leftrightarrow & \lim\limits_{x\to a}\Big(\varphi(x)\psi(x) - L_1\psi(x) - L_2\varphi(x) + L_1L_2\Big) & = & \underbrace{\lim\limits_{x\to a}(\varphi(x) - L_1)(\psi(x) - L_2)}_{0} \\ \Leftrightarrow & \lim\limits_{x\to a}(\varphi(x)\psi(x)) - L_1\underbrace{\lim\limits_{x\to a}(\psi(x))}_{L_2} - L_2\underbrace{\lim\limits_{x\to a}(\varphi(x))}_{L_1} + \lim\limits_{x\to a}(L_1L_2) & = & 0 \\ \Leftrightarrow & \lim\limits_{x\to a}(\varphi(x)\psi(x)) & = & \boxed{L_1L_2} \end{array}$$

□

Remark 6. *As the proof is rather elementary, we've assumed that for a constant $k \in \mathbb{R}$,*

$$\lim_{x \to a}(k\varphi(x)) = k \lim_{x \to a}(\varphi(x))$$

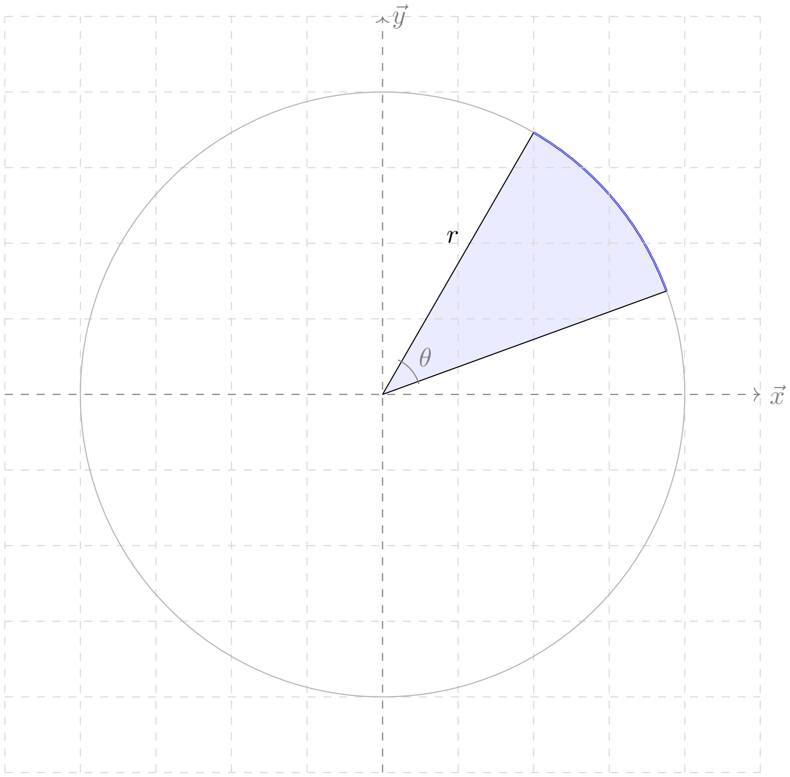
We've also assumed that the product of two absolute values is the absolute values of the product. Again, the proof is elementary.

So, what we did was applying the definition of the derivative on $x \mapsto \sin x$; this yields the limit of a sum. We know we can split it into sums of limits, provided the two individual limits exist. So we must now try to compute those two limits.

We'll first need to define what a *circular sector* is, and how to express its *area*, from which, you'll see, computing the area of a circle is but a special case. We will also need to establish another important result on limits: the *squeeze theorem*.

Let's start with circular sector:

Definition 2 (circular arc, circular sector)**.** *A circular arc is a portion of the circle between two points of that circle. A circular sector is a portion of disk enclosed between two (usually distinct) radii*[2] *and a circular arc.*



Remark 7. *A circle then, is but a circular sector of angle 2π.*

Unfortunately, if we want to compute the area of a circular sector, we either need some integral calculus to compute it directly, or indirectly, via the area of a circle. In addition to the definition of an integral, to compute the actual integral representing the area of a circle (or that of a sector), we will need the u-substitution rule, which demands the fundamental theorem of calculus.

A rigorous, extensive treatment of Riemann integration would require more work than what we aim to achieve here; you may want to refer to a full real analysis course for more.

[2]"radii" is the plural of "radius"

Definition 3 ((Riemann) integral)**.** *TODO*

Theorem 9 (Fundamental theorem of calculus)**.** *TODO*

Proof. TODO □

Theorem 10 (u-substitution/reversed chain-rule (single variable))**.** *Let* $U \subseteq \mathbb{R}$*,* $(a, b) \in \mathbb{R}^2$*. Let* $\varphi : U \to \mathbb{R}$ *and* $\nu : [a, b] \to U$ *be respectively* C^0 *and* C^1 *functions. Then,*

$$\boxed{\int_a^b \varphi(\nu(x))\nu'(x)\,dx = \int_{\nu(a)}^{\nu(b)} \varphi(u)\,du}$$

Proof. Let's assume here that continuous functions are integrable. Then, the existence of both integrals is guaranteed by the restrictions imposed on φ and ν.

Then, let us note Φ the anti-derivative of φ, i.e. $\Phi' = \varphi$, which by the same assumption as before must exists, since φ is continuous. Let us then apply the chain-rule to $\Phi \circ \nu$:

$$(\Phi \circ \nu)'(x) = \nu'(x)\Phi'(\nu(x)) = \varphi(\nu(x))\nu'(x)$$

Finally, by integrating both sides and repeatedly applying the fundamental theorem of calculus twice, we obtain:

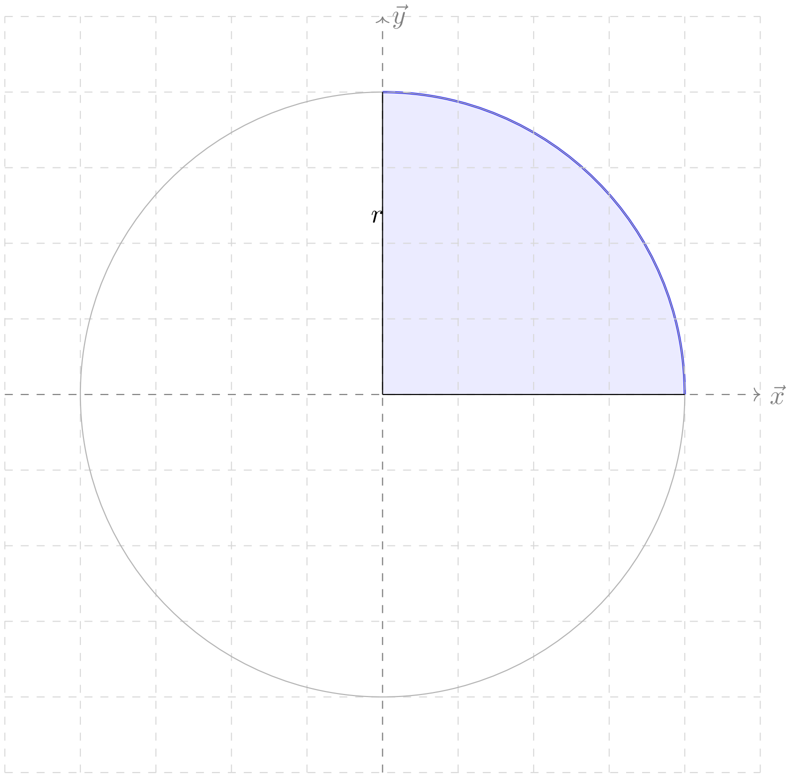
$$\begin{aligned}\int_a^b \varphi(\nu(x))\nu'(x)\,dx =& \int_a^b (\Phi \circ \nu)'(x)dx \\ =&(\Phi \circ \nu)(b) - (\Phi \circ \nu)(a) \\ =&\Phi(\nu(b)) - \Phi(\nu(a)) \\ =& \int_{\nu(a)}^{\nu(b)} \varphi(u)\,du\end{aligned}$$

□

Theorem 11 (circle area)**.** *The area of a circle of radius* r *is given by:*

$$\boxed{\pi r^2}$$

Proof. The problem of the area of a circle can be reduced via symmetry to the integration of a positive curve on an interval. More precisely, let's consider a circle centered at the origin of radius r. To compute its area, suffice to compute the area of a quadrant of it, say the first quadrant (the blueish one; starting from this one, the four quadrant are enumerated following an anti-clockwise/trigonometric direction)



By symmetry, the area of a circle A_S is four times that of any quadrant A_{Q_i}. Furthermore, the first quadrant's area A_{Q_1} is given by the area under the curve describing a circle that we saw earlier in 4, restricted to $[0, r]$. More precisely:

$$A_S = 4A_{Q_i} = 4A_{Q_1} = 4\int_0^r \sqrt{r^2 - x^2}\,dx$$

This integral can be considered in the form:

$$\int_{\nu(\theta_0)}^{\nu(\theta_1)} \varphi(x)\,dx$$

Let's perform a first change of variable by setting:

$$\nu(\theta) = r\sin\theta; \qquad \nu'(\theta) = r\cos\theta$$

It follows that:

$$\nu(\theta_0) = 0 \Rightarrow \theta_0 = 0; \qquad \nu(\theta_1) = 0 \Rightarrow \theta_1 = \frac{\pi}{2}$$

The integral can now be rewritten:

$$\begin{aligned}
A_S &= 4\int_0^{\pi/2} \phi(\nu(\theta))\nu'(\theta)\,d\theta \\
&= 4\int_0^{\pi/2} \sqrt{r^2 - (r\sin\theta)^2} r\cos\theta\,d\theta \\
&= 4r^2\int_0^{\pi/2} \sqrt{1 - \sin^2\theta}\cos\theta\,d\theta \qquad (r > 0) \\
&= 4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta \qquad (\cos\theta > 0,\ \text{Pythagorean theorem})
\end{aligned}$$

However, with (5) we have:

$$4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta \qquad = \qquad 4r^2\int_0^{\pi/2} (\sin(\theta + \frac{\pi}{2}))^2\,d\theta$$

We can perform a second change of variable on the rightmost integral:

$$\mu(u) = \frac{\pi}{2} - u; \qquad \mu'(u) = -1$$

Then:

$$\mu(u_0) = 0 \Rightarrow u_0 = \frac{\pi}{2}; \qquad \mu(u_1) = \frac{\pi}{2} \Rightarrow u_1 = 0$$

The integral becomes:

$$4r^2 \int_0^{\pi/2} (\sin(\theta + \frac{\pi}{2}))^2 \, d\theta \quad = \quad -4r^2 \int_{\pi/2}^{0} \sin(u)^2 \, du \quad = \quad 4r^2 \int_0^{\pi/2} \sin^2\theta \, d\theta$$

Referring back to our previous integral equality, this means we have:

$$S_A \quad = \quad 4r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta \quad = \quad 4r^2 \int_0^{\pi/2} \sin^2\theta \, d\theta$$

However, by linearity of integration, and by the Pythagorean theorem, again, we have

$$\int_0^{\pi/2} \cos^2\theta \, d\theta + \int_0^{\pi/2} \sin^2\theta \, d\theta = \int_0^{\pi/2} \underbrace{\cos^2\theta + \sin^2\theta}_{=1} \, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2} = 2\int_0^{\pi/2} \cos^2\theta \, d\theta$$

Hence,

$$S_A = 2r^2 \underbrace{2\int_0^{\pi/2} \cos^2\theta \, d\theta)}_{=\pi/2} = \boxed{\pi r^2}$$

□

Theorem 12 (circular sector area)**.** *The area of a circular sector, of angle θ, in a circle of radius r is given by:*

$$\boxed{\frac{1}{2} r^2 \theta}$$

Proof. Because the area is evenly distributed on a circle, this is a simple cross-multiplication[3] involving the area of a circle:

$$\begin{aligned} 2\pi &\rightarrow \pi r^2 \\ \theta &\rightarrow A_\theta = \frac{\theta \pi r^2}{2\pi} = \boxed{\frac{1}{2} r^2 \theta} \end{aligned}$$

□

Remark 8. *We could also have proved it directly with an integral, as we did for the circle.*

Theorem 13 (squeeze theorem)**.** *Let φ, ψ, ϕ be three real-valued functions defined on an interval $I \subset \mathbb{R}$, and a be a point of I. If, $(\forall x \in I/\{a\})$ ($x \in I$ but $x \neq a$), we have:*

$$\varphi(x) \leq \psi(x) \leq \phi(x)$$

With:

$$\lim_{x \to a} \varphi(x) = \lambda = \lim_{x \to a} \phi(x)$$

Then:

$$\boxed{\lim_{x \to a} \psi(x) = \lambda}$$

[3] https://en.wikipedia.org/wiki/Cross-multiplication

Proof. Let's translate our limits into their $(\epsilon - \delta)$ form, considering a single ϵ

$$(\forall \epsilon > 0), (\exists \delta_1 \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta_1) \Rightarrow (|\varphi(x) - \lambda| < \epsilon)$$

$$(\forall \epsilon > 0), (\exists \delta_2 \in \mathbb{R}), (\forall x \in \mathbb{R}) (|x - a| < \delta_2) \Rightarrow (|\phi(x) - \lambda| < \epsilon)$$

So, let $\epsilon > 0$, and let $\delta = \min(\delta_1, \delta_2)$. For the same reasons as with previous limits proofs, this δ implies both the inequalities involving ϵ, which can be restated, by definition of the absolute value, as:

$$-\epsilon < \varphi(x) - \lambda < \epsilon$$
$$-\epsilon < \phi(x) - \lambda < \epsilon$$

But we also have:

$$\varphi(x) \leq \psi(x) \leq \phi(x)$$
$$\Leftrightarrow \varphi(x) - \lambda \leq \psi(x) - \lambda \leq \phi(x) - \lambda$$

So:

$$-\epsilon < \varphi(x) - \lambda \leq \psi(x) - \lambda \leq \phi(x) - \lambda < \epsilon$$
$$\Leftrightarrow |\psi(x) - \lambda| < \epsilon$$

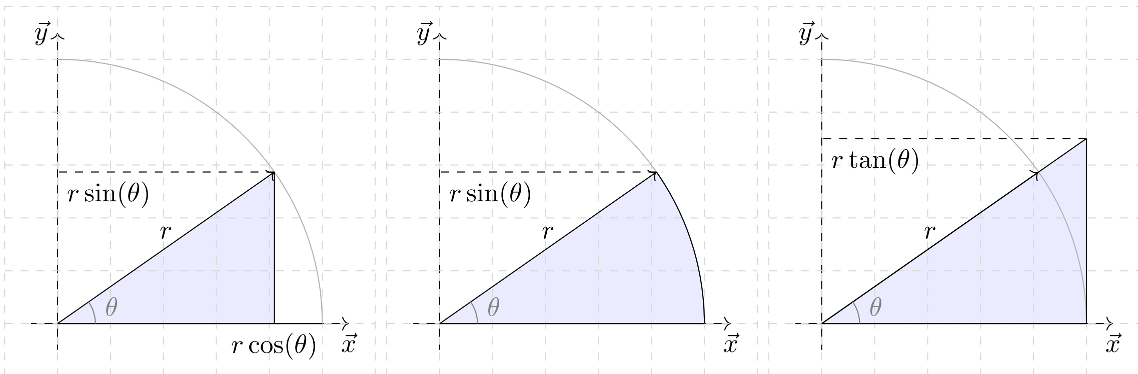
Which concludes the proof, as we now have an $(\epsilon - \delta)$ statement on ψ, equivalent to a limit.
□

We now have everything we need to start computing the limits involved in the differentiation of sine:

Theorem 14.

$$\boxed{\lim_{\epsilon \to 0} \frac{\sin \epsilon}{\epsilon} = 1}$$

Proof. Consider the three following blueish areas:



The three areas are definitely ordered from smaller to bigger (left to right), and we can also determine them: the middle one is that of a sector, while the two side ones are right triangles (so their areas is half of the corresponding rectangle). We then have the following inequalities:

$$\begin{array}{rrrr} & \frac{1}{2}r^2 \cos\theta \sin\theta \leq & \frac{1}{2}r^2\theta \leq & \frac{1}{2}r^2 \tan\theta \\ \Leftrightarrow & \cos\theta\sin\theta \leq & \theta \leq & \frac{\sin\theta}{\cos\theta} \\ \Leftrightarrow & \cos\theta \leq & \frac{\theta}{\sin\theta} \leq & \frac{1}{\cos\theta} \\ \Leftrightarrow & \frac{1}{\cos\theta} \geq & \frac{\sin\theta}{\theta} \geq & \cos\theta \end{array}$$

But, as θ goes to zero, the two extremes of this inequalities become:

$$\lim_{\theta \to 0} \frac{1}{\cos\theta} = 1; \qquad \lim_{\theta \to 0} \cos\theta = 1$$

Hence by the *squeeze theorem*, it follows that we *must* have:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$$

□

As for the other limit:

Theorem 15.

$$\boxed{\lim_{\epsilon\to 0}\frac{\cos\epsilon-1}{\epsilon}=0}$$

Proof. We will rely on the previous limit; this will be a "proof with a trick" (multiplying by $a/a=1$; in the present context, a will always be non-zero). Note also at the end that we can apply the product rules for limits given the existence of both sublimits.

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{\cos\epsilon-1}{\epsilon} &= \lim_{\epsilon\to 0}\left(\frac{\cos\epsilon-1}{\epsilon}\times\frac{\cos\epsilon+1)}{\cos\epsilon+1)}\right)\\
&= \lim_{\epsilon\to 0}\left(\frac{\cos^2\epsilon-1}{\epsilon(\cos\epsilon+1))}\right)\\
&= -\lim_{\epsilon\to 0}\left(\frac{\sin^2\epsilon}{\epsilon(\cos\epsilon+1))}\right)\\
&= -\lim_{\epsilon\to 0}\left(\frac{\sin\epsilon}{\epsilon}\times\frac{\sin\epsilon}{\cos\epsilon+1}\right)\\
&= -\underbrace{\lim_{\epsilon\to 0}(\frac{\sin\epsilon}{\epsilon})}_{\to 1}\times\underbrace{\lim_{\epsilon\to 0}(\frac{\sin\epsilon}{\cos\epsilon+1})}_{\to 0/2=0}\\
&= \boxed{0}
\end{aligned}$$

□

We now have everything to conclude: let's recapitulate all the intermediate steps to compute $\sin'$:

Theorem 16 (sine derivative)**.**

$$(\forall x\in\mathbb{R}),\quad \boxed{\sin'(x)=\cos(x)}$$

Proof.

$$\begin{aligned}
(\forall x\in\mathbb{R}),\quad \sin'(x) &\triangleq \lim_{\epsilon\to 0}\frac{\sin(x+\epsilon)-\sin x}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{\sin x\cos\epsilon+\cos x\sin\epsilon-\sin x}{\epsilon}\\
&= \lim_{\epsilon\to 0}\left(\frac{\sin x(\cos\epsilon-1)}{\epsilon}+\frac{\cos x\sin\epsilon}{\epsilon}\right)\\
&= \sin x\underbrace{\lim_{\epsilon\to 0}\frac{\cos\epsilon-1}{\epsilon}}_{=0}+\cos x\underbrace{\lim_{\epsilon\to 0}\frac{\sin\epsilon}{\epsilon}}_{=1}\\
&= \boxed{\cos x}
\end{aligned}$$

□

$d\cos t/dt$

Theorem 17 (cosine derivative)**.**

$$(\forall x\in\mathbb{R}),\quad \boxed{\cos'(x)=-\sin(x)}$$

Proof. The results follow from the shifts formulas **??**

$$\begin{aligned}(\forall x \in \mathbb{R}), \quad \cos'(x) &= \sin'(x + \frac{\pi}{2}) \\ &= (\sin \circ (y \mapsto y + \frac{\pi}{2}))'(x) \\ &= \cos(x + \frac{\pi}{2}) \\ &= \boxed{-\sin(x)}\end{aligned}$$

□

---

de^t/dt

This one, as mentioned in the book, is "trivial" when we define the exponential function to be precisely the function which is equal to its derivative (and such as $e^0 = 1$).

And this is usually the way the exponential function will be first introduced to students. You may want to have a look at other equivalent characterization of the function[4]. Trying to compute an exponential defined on a development in infinite series carries a certain aesthetic for instance.

$$\boxed{\frac{d}{dt}e^t \triangleq e^t}$$

---

d ln t/dt

As for the exponential, there can be some variety here depending on how we *characterize* the ln function [5]. Usually, it will be introduced as the *inverse function* of the exponential:

Definition 4 (natural logarithm)**.** *The natural logarithm function is defined as the function* ln *such that:*

$$(\forall x \in \mathbb{R}), \quad \boxed{e^{\ln(x)} = x}$$

Remark 9. *To rigorously establish this definition, would have needed to prove that the exponential is invertible.*

Theorem 18 (natural logarithm derivative)**.**

$$(\forall x \in \mathbb{R}), \quad \boxed{\ln'(x) = \frac{1}{x}}$$

Proof. The proof develops from the previous definition of the logarithm by integrating both side and then applying the chain rule:

$$\begin{aligned} & (\forall x \in \mathbb{R}), \quad e^{\ln(x)} &= x \\ \Leftrightarrow \quad & \frac{d}{dx}e^{\ln(x)} &= \frac{d}{dx}x \\ \Leftrightarrow \quad & \ln'(x)\underbrace{e^{\ln(x)}}_{=x} &= 1 \\ \Leftrightarrow \quad & \ln'(x) &= \boxed{\frac{1}{x}}\end{aligned}$$

□

Remark 10. *For the sake of completeness, some authors[6], will for instance start by defining the logarithm as an integral, and then define the exponential as the inverse of the logarithm. From which they can prove that the derivative of the exponential is the exponential.*

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[4] `https://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function`
[5] `https://en.wikipedia.org/wiki/Natural_logarithm#Definitions`
[6] `https://www.whitman.edu/mathematics/calculus_late_online/section09.02.html`