The Theoretical Minimum

Classical Mechanics - Solutions

L02E07

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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May 10, 2023

Exercise 1. *Show that the position and velocity vectors from Eq.s* (3) *are orthogonal.*

Actually, Eq.s (3) refers to velocity and acceleration, not position. Because of this ambiguity, let's look for which pair of vectors are orthogonal among the three. Let's start by recalling how they have been defined (knowing that velocity and acceleration are obtained by differentiating position respectively once and twice):

$$\begin{aligned} r_x(t) &= R\cos(\omega t); & r_y(t) &= R\sin(\omega t) \\ v_x(t) &= -R\omega\sin(\omega t); & v_y(t) &= R\omega\cos(\omega t) \\ a_x(t) &= -R\omega^2\cos(\omega t); & a_y(t) &= -R\omega^2\sin(\omega t) \end{aligned}$$

We've established in I01E06 that two vectors are orthogonal if their dot product is zero; where the dot product has been defined as:

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{v} &= u_xv_x + u_yv_y + u_zv_z \\ &= \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta_{uv} \end{aligned}$$

Then, let's compute a few dot products (we're in the plane, so the z components must be zero):

$$\begin{aligned} \boldsymbol{r}\cdot\boldsymbol{v} &= r_xv_x + r_yv_y \\ &= R\cos(\omega t)\times(-R\omega\sin(\omega t)) + R\sin(\omega t)\times R\omega\cos(\omega t) \\ &= \boxed{0} \\ \boldsymbol{r}\cdot\boldsymbol{a} &= r_xa_x + r_ya_y \\ &= R\cos(\omega t)\times(-R\omega^2\cos(\omega t)) + R\sin(\omega t)\times(-R\omega^2\sin(\omega t)) \\ &= -R^2\omega^2\underbrace{(\cos^2(\omega t)+\sin^2(\omega t))}_{=1} \\ &= \boxed{-(R\omega)^2 \neq 0} \\ \boldsymbol{v}\cdot\boldsymbol{a} &= v_xa_x + v_ya_y \\ &= (-R\omega\sin(\omega t))\times(-R\omega^2\cos(\omega t)) + R\omega\cos(\omega t)\times(-R\omega^2\sin(\omega t)) \\ &= \boxed{0} \end{aligned}$$

Hence both position and acceleration are orthogonal with velocity.

Remark 1. *Regarding $\boldsymbol{r}\cdot\boldsymbol{a}$, we could also have observed that $\boldsymbol{a} = -\omega^2\boldsymbol{r}$: the vectors are collinear, so they simply can't be orthogonal. From there, were we to already have established $\boldsymbol{r}\cdot\boldsymbol{v} = 0$, we could have inferred $\boldsymbol{v}\cdot\boldsymbol{a} = -\omega^2\boldsymbol{v}\cdot\boldsymbol{r} = 0$, using the (bi)linearity of the dot product.*