The Theoretical Minimum

Classical Mechanics - Solutions

L02E08

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Calculate the velocity, speed and acceleration for each of the following position vectors. If you have graphing software, plot each position vector, each velocity vector, and each acceleration vector.*

$$
\begin{aligned}
\vec{r} &= (\cos\omega t, e^{\omega t}) \\
\vec{r} &= (\cos(\omega t - \phi), \sin(\omega t - \phi)) \\
\vec{r} &= (c\cos^3 t, c\sin^3 t) \\
\vec{r} &= (c(t - \sin t), c(1 - \cos t))
\end{aligned}
$$

Let's recall that each component of the velocity and acceleration vectors are defined respectively as the derivative and second derivative of the corresponding component of the position vector:

$$
\begin{aligned}
\boldsymbol{r}(t) &= (x(t), y(t)) \\
\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) &= (\dot{x}(t), \dot{y}(t)) \\
\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) &= (\ddot{x}(t), \ddot{y}(t))
\end{aligned}
$$

So this is just a differentiation exercise in disguise. We'll be using a fast pace here (80% of the work is about applying the chain rule); if you need a slower approach, see for instance L02E01, where we go in-depth on how to apply common differentiation rules.

$\boldsymbol{r_0(t) = (\cos(\omega t), e^{\omega t})}$

$$
\begin{aligned}
\boldsymbol{r_0}(t) &= (\cos(\omega t), e^{\omega t}) \\
\boldsymbol{v_0}(t) = \dot{\boldsymbol{r_0}}(t) &= \boxed{(-\omega\sin(\omega t), \omega e^{\omega t})} \\
\boldsymbol{a_0}(t) = \dot{\boldsymbol{v_0}}(t) = \ddot{\boldsymbol{r_0}}(t) &= \boxed{(-\omega^2\cos(\omega t), \omega^2 e^{\omega t})}
\end{aligned}
$$

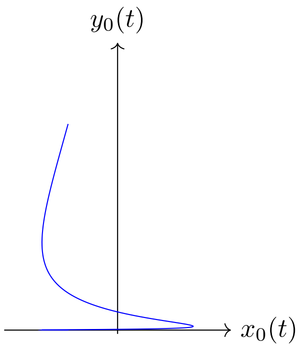

Figure 1: $\omega = 1$; $\boldsymbol{r}_0(t) = (\cos(\omega t), e^{\omega t})$

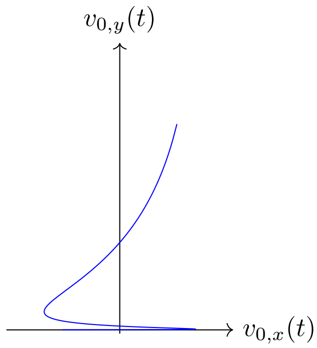

Figure 2: $\omega = 1$; $\boldsymbol{v}_0(t) = (-\omega\sin(\omega t), \omega e^{\omega t})$

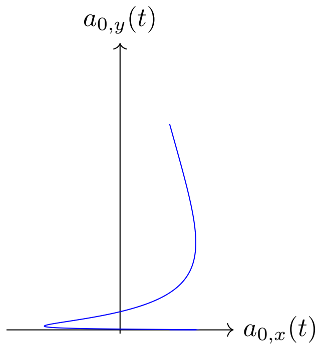

Figure 3: $\omega = 1$; $\boldsymbol{a}_0(t) = (-\omega^2\cos(\omega t), \omega^2 e^{\omega t})$

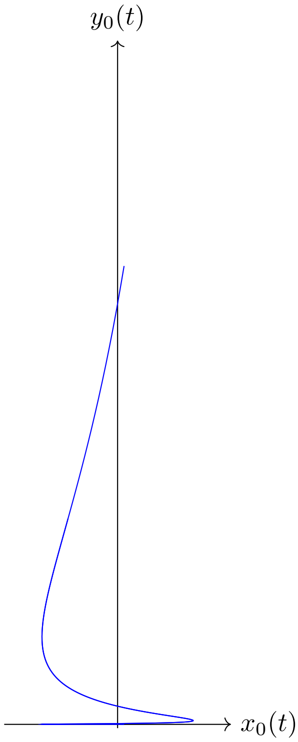

Figure 4: $\omega = 1.2$;
$\boldsymbol{r}_0(t) = (\cos(\omega t), e^{\omega t})$

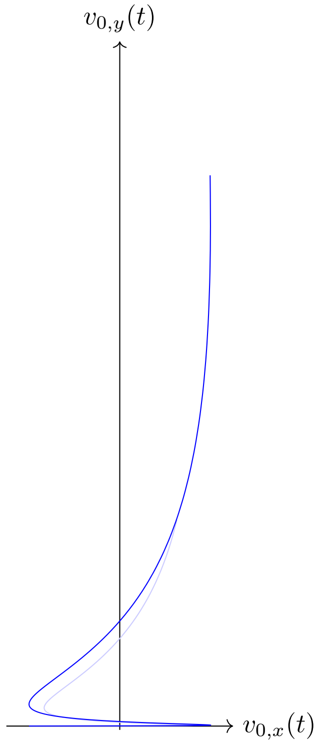

Figure 5: $\omega = 1.2$;
$\boldsymbol{v}_0(t) = (-\omega \sin(\omega t), \omega e^{\omega t})$

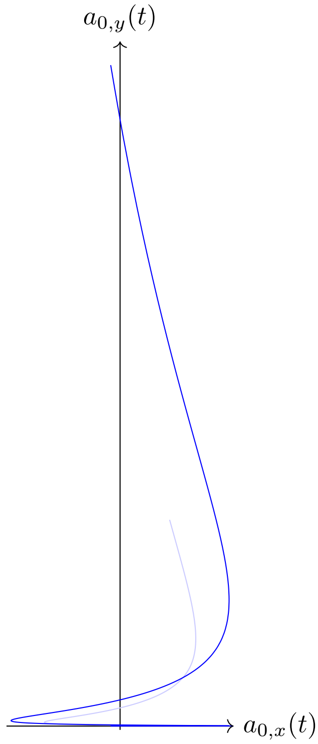

Figure 6: $\omega = 1.2$;
$\boldsymbol{a}_0(t) = (-\omega^2 \cos(\omega t), \omega^2 e^{\omega t})$

Remark 1. *Increasing ω will:*

- *For the position, increase the distance at which we travel in the y direction; the distance in the x direction will be the same, because it's constrained by a* $\cos$*, but we'll get there faster:*
- *For the velocity, we will go faster in both the x and y directions; we've plotted in a fainted blue on the second graph the $\boldsymbol{v}_0(t)$ for $\omega = 1$ for comparison, because the effect in the x direction is small;*
- *And obviously if the velocity increases, the acceleration must increase accordingly, which it does, quadratically, both in the x and y directions (again, we've plotted in a fainted blue on the second graph $\boldsymbol{a}_0(t)$ for $\omega = 1$ for comparison).*

$\boldsymbol{r_1(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))}$

$$
\begin{aligned}
\boldsymbol{r_1}(t) &= (\cos(\omega t - \phi), \sin(\omega t - \phi)) \\
\boldsymbol{v_1}(t) = \dot{\boldsymbol{r_1}}(t) &= \boxed{(-\omega \sin(\omega t - \phi), \omega \cos(\omega t - \phi))} \\
\boldsymbol{a_1}(t) = \dot{\boldsymbol{v_1}}(t) = \ddot{\boldsymbol{r_1}}(t) &= \boxed{(-\omega^2 \cos(\omega t - \phi), -\omega^2 \sin(\omega t - \phi))} \quad = -\omega^2 \boldsymbol{r}_1(t)
\end{aligned}
$$

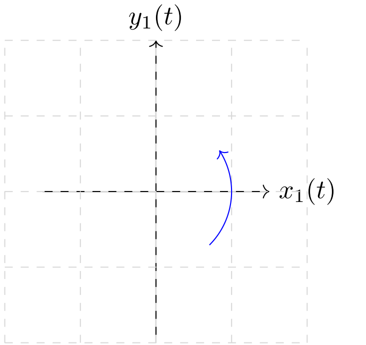

Figure 7: $\omega = 1, \varphi = 0$; $\boldsymbol{r}_1(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

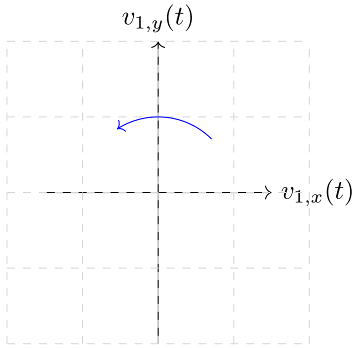

Figure 8: $\omega = 1, \varphi = 0$; $\boldsymbol{v}_1(t) = \omega(-\sin(\omega t - \phi), \cos(\omega t - \phi))$

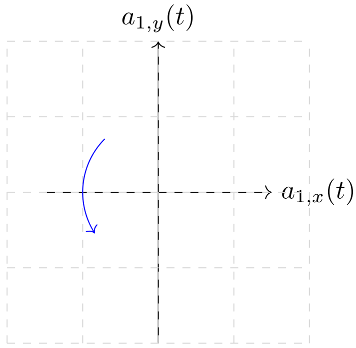

Figure 9: $\omega = 1, \varphi = 0$; $\boldsymbol{a}_1(t) = -\omega^2 \boldsymbol{r}_1(t))$

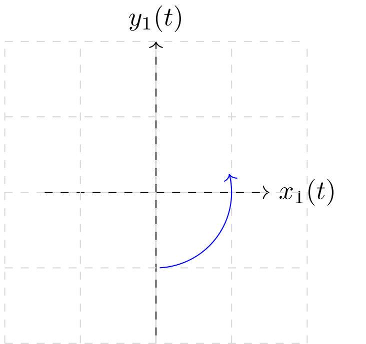

Figure 10: $\omega = 1.3, \varphi = 0.5$; $\boldsymbol{r}_1(t) = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

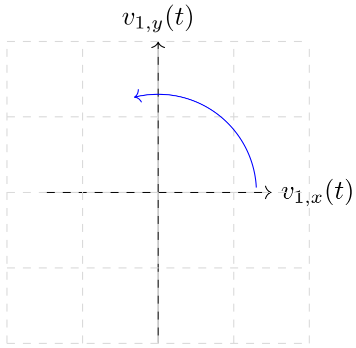

Figure 11: $\omega = 1.3, \varphi = 0.5$; $\boldsymbol{v}_1(t) = \omega(-\sin(\omega t - \phi), \cos(\omega t - \phi))$

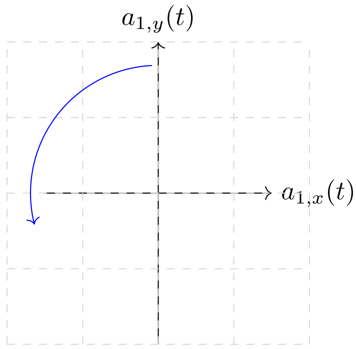

Figure 12: $\omega = 1.3, \varphi = 0.5$; $\boldsymbol{a}_1(t) = -\omega^2 \boldsymbol{r}_1(t))$

Remark 2. *All those plots were made with $t \in [-\pi/4, \pi/3]$ so as to make more visible the effect of changing the phase ϕ, which only alters our starting/ending point. The alteration would have been hidden were t to have gone through an interval wider or equal than 2π. An arrow has been added to indicate the ending point.*

ω is the angular velocity, or the number of radians the particle move per unit of time. Naturally, if it's increased, the particle will go further, faster; the increase in speed will demand a corresponding increase in acceleration.

$\boldsymbol{r_2(t) = c(\cos t)^3, c(\sin t)^3)}$

$\boldsymbol{a_2}$ is the most complex derivative for this exercise. We start by applying the product rule ($uv = u'v + uv'$), and the chain rule on one of the resulting factor.

$$\begin{aligned}
\boldsymbol{r_2}(t) &= (c\cos^3 t, c\sin^3 t) \\
\boldsymbol{v_2}(t) = \dot{\boldsymbol{r_2}}(t) &= \boxed{3c(-\sin t\cos^2 t, \cos t\sin^2 t)} \\
\boldsymbol{a_2}(t) = \dot{\boldsymbol{v_2}}(t) = \ddot{\boldsymbol{r_2}}(t) &= 3c(-\cos t\cos^2 t + (-\sin t)(-\sin t)(2\cos t), -\sin t\sin^2 t + \cos t\cos t 2\sin t) \\
&= \boxed{3c((\cos t)(2\sin^2 t - \cos^2 t), (\sin t)(2\cos^2 t - \sin^2 t))}
\end{aligned}$$

Remark 3. *We may be able to simplify the expression of the acceleration $\boldsymbol{a}_2$.*

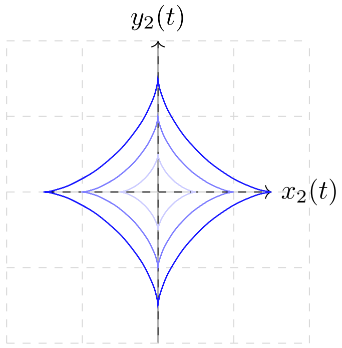

Figure 13: $c = 0.5,\ 1,\ 1.5$;$\boldsymbol{r}_2(t) = (c\cos^3 t, c\sin^3 t)$

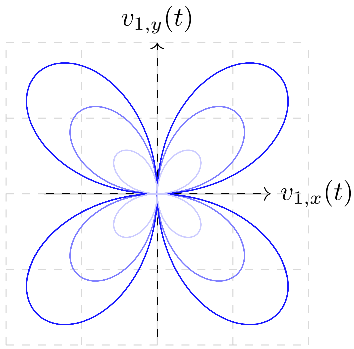

Figure 14: $c = 0.5,\ 1,\ 1.5$; $\boldsymbol{v}_2(t) = 3c(-\sin t\cos^2 t, \cos t\sin^2 t)$

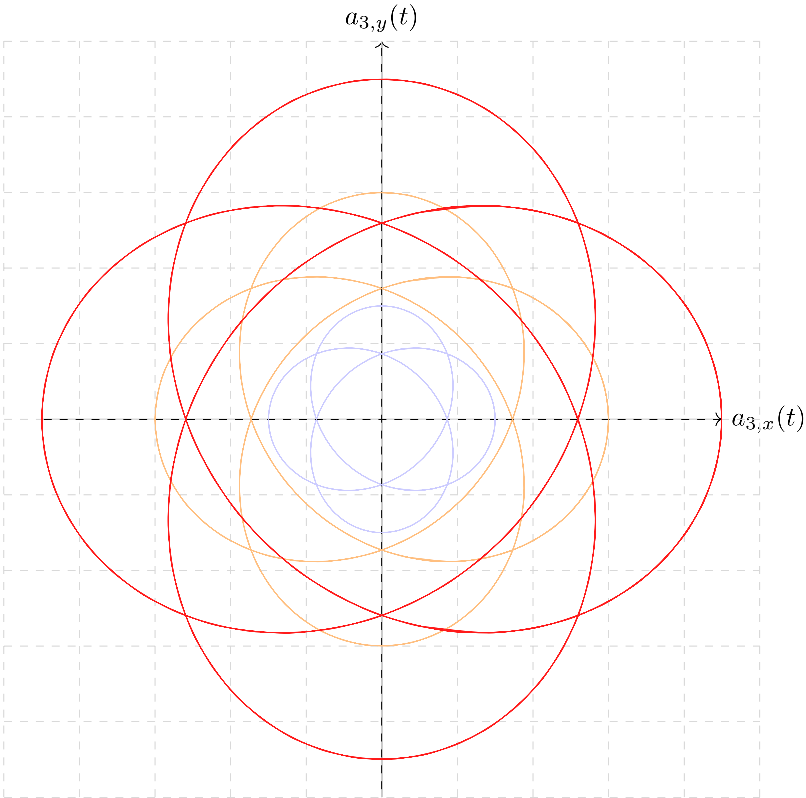

Figure 15: $c = 0.5,\ 1,\ 1.5$ (blue, orange, red); $\boldsymbol{a}_2(t) = 3c((\cos t)(2\sin^2 t - \cos^2 t), (\sin t)(2\cos^2 t - \sin^2 t))$

Remark 4. *We see from the equations that c is a scaling factor, operating on both axes. If we increase it, we will go higher (y-axis) and further away (x-axis) in the same amount of time t, hence we'll need greater speed, in both axis, and greater acceleration, again on both axes.*

$\boldsymbol{r_3(t) = (c(t - \sin t), c(1 - \cos t))}$

$$\begin{aligned}
\boldsymbol{r_3}(t) &= (c(t - \sin t), c(1 - \cos t)) \\
\boldsymbol{v_3}(t) = \dot{\boldsymbol{r_3}}(t) &= \boxed{c(1 - \cos t, \sin t)} \\
\boldsymbol{a_3}(t) = \dot{\boldsymbol{v_3}}(t) = \ddot{\boldsymbol{r_3}}(t) &= \boxed{c(\sin t, \cos t)}
\end{aligned}$$

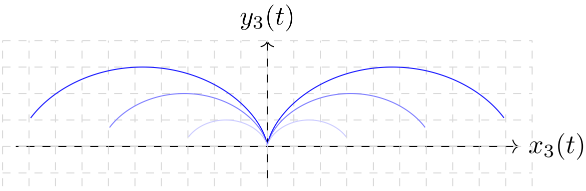

Figure 16:
$c = 0.5,\ 1,\ 1.5$;
$\boldsymbol{r}_3(t) = (c(t - \sin t), c(1 - \cos t))$

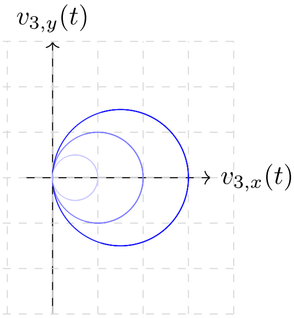

Figure 17:
$c = 0.5,\ 1,\ 1.5$;
$\boldsymbol{v}_3(t) = c(1 - \cos t, \sin t)$

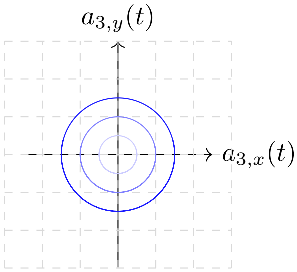

Figure 18:
$c = 0.5,\ 1,\ 1.5$;
$\boldsymbol{a}_3(t) = c(\sin t, \cos t)$

Remark 5. *As for the previous exercise, c is a scaling factor, with the same kind of impact as before.*