The Theoretical Minimum

Classical Mechanics - Solutions

L05E01

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Exercise 1. *Prove Eq. (3). Hint: Use the product rule for differentiation.*

This is a very simple differentiation exercise. I'm going to be very slow, because using a "normal" pace would essentially mean stating the result, as it's been done in the book, hence negating the need for this exercise. If you can't compute this kind of derivative in your head at this stage, I would recommend practising until it becomes second nature. For instance, *Paul's Online Notes*[1] has plenty of corrected exercises on differentiation.

Let's recall the context a little bit: we're in the case of a single particle moving along the $\vec{x} - axis$ under the influence of a force $F(x)$. The *kinetic energy* of that particle, that is, the energy that the particle has because of its motion, is noted T and is "defined" as:

$$T = \frac{1}{2}mv^2$$

Eq. (3) refers to the time derivative of v^2, in the context of its usage in the formulation of kinetic energy. There are (at least) two ways to evaluate this derivative; let's start by using the authors' hint regarding the product rule; let's make the time dependency more obvious:

$$\frac{d}{dt}v(t)^2 = \frac{d}{dt}v(t)v(t)$$

Now let's recall the *product rule* is, for φ and ψ two real-valued functions of t:

$$(\varphi\psi)' = \varphi'\psi + \varphi\psi'$$

Hence, in the case of $\varphi = \psi = v$, we have:

$$\begin{aligned}
\frac{d}{dt}v(t)^2 &= \frac{d}{dt}v(t)v(t) \\
&= \dot{v}(t)v(t) + v(t)\dot{v}(t) \\
&= \boxed{2v\dot{v}(t)} \quad \square
\end{aligned}$$

Remark 1. *We could have also used the chain rule; for φ a function of t, and ψ and function whose domain (input) is the codomain (output) of φ:*

$$(\psi \circ \varphi)'(t) = (\psi(\varphi(t))' = \varphi'(t)\psi'(\varphi(t))$$

In this case, $\psi = (x \mapsto x^2)$ and $\varphi = v$, so:

$$\begin{aligned}
\frac{d}{dt}v(t)^2 &= \frac{d}{dt}(x \mapsto x^2)(v(t)) \\
&= \dot{v}(t)(x \mapsto x^2)'(v(t)) \\
&= \dot{v}(t)(x \mapsto 2x)(v(t)) \\
&= \boxed{2v\dot{v}(t)} \quad \square
\end{aligned}$$

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[1] https://tutorial.math.lamar.edu/Problems/CalcI/DerivativeIntro.aspx

Remark 2. *As a reminder, both the product rule and the chain rule have been proved in L02E04.*