The Theoretical Minimum

Classical Mechanics - Solutions

L05E02

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Exercise 1. *Consider a particle in two dimensions, x and y. The particle has mass m. The potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. Work out the equations of motion. Show that there are circular orbits and that all orbits have the same period. Prove explicitly that the total energy is conserved.*

Equations of motion

For this system, the potential energy V is:

$$V = \frac{1}{2}k(x^2 + y^2) \tag{1}$$

By Newton's second law of motion, given $\boldsymbol{r} = (x, y)$, we have:

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{r}} \tag{2}$$

Or,

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \end{aligned} \tag{3}$$

We know by equation (5) of this lecture that to each coordinate x_i of the configuration space $\{x\}$, there is a force F_i, derived from the potential energy V:

$$F_i(\{x\}) = -\frac{\partial}{\partial x_i}V(\{x\}) \tag{4}$$

Which in our case, translates to:

$$\begin{aligned} F_x(x, y) &= -\frac{\partial}{\partial x}V(x, y) = -kx \\ F_y(x, y) &= -\frac{\partial}{\partial y}V(x, y) = -ky \end{aligned} \tag{5}$$

Combining (3) and (5):

$$\begin{aligned} m\ddot{x}(t) &= -kx(t) \\ m\ddot{y}(t) &= -ky(t) \end{aligned} \tag{6}$$

Which we known from L03E04: to be differential equations associated to harmonic motion, and (mathematically) solved by a slightly more general solution that the one proposed in L03E04:

$$\begin{aligned} x(t) &= \alpha_x \cos(\omega t - \theta_x) + \beta_x \sin(\omega t - \theta_x) \\ y(t) &= \alpha_y \cos(\omega t - \theta_y) + \beta_y \sin(\omega t - \theta_y) \\ \omega^2 &= \frac{k}{m} \end{aligned} \tag{7}$$

Indeed, considering e.g. $x(t)$, with simplified variable names:

$$
\begin{aligned}
v(t) = \quad \dot{x}(t) = \quad & \omega(-\alpha\sin(\omega t - \theta) + \beta\cos(\omega t - \theta)) \\
a(t) = \quad \ddot{x}(t) = \quad & -\omega^2(\alpha\cos(\omega t - \theta) + \beta\sin(\omega t - \theta)) \\
= \quad & -\omega^2 x(t)
\end{aligned}
\tag{8}
$$

Where, to differentiate e.g. $\alpha\cos(\omega t - \theta)$, we define $\phi(\omega) = \omega t - \theta$, so as to use the chain rule for derivation:

$$
\frac{d}{dx}f(g(x)) = g'(x)f'(g(x))
\tag{9}
$$

Note that, in this case, as already suggested on L03E04 $\alpha_{x,y}$ and $\beta_{x,y}$ can be determined from the initial position and velocity, that is, from $x(t=0), \dot{x}(t=0), y(t=0), \dot{y}(t=0)$. For instance, assuming $\theta_{x,y} = 0$ to simplify the calculus:

$$
\begin{aligned}
x(0) &= \alpha_x\cos(0) + \beta_x\sin(0) \\
&= \alpha \\
\dot{x}(0) &= \omega(-\alpha\sin(0) + \beta\cos(0)) \\
&= \omega\beta \\
&= \sqrt{\frac{k}{m}}\beta
\end{aligned}
\tag{10}
$$

Circular orbits

Remark 1. *This exercise was completed a few months before writing the present remark. The approach taken is mathematically interesting, but clumsy, from a physicist's point of view. Given the question, and given the fact that the simplest circular orbit has already been studied at the end of Chapter 2 (Motion), it would have been wiser to start by verifying this motion, rather than working from the most general solution to the involved differential equation.*

The quest for the general solution to the differential equation itself could and should, in the context of a physics exercise, have been polarized by a wiser choice of referential, to reduce the complexity of the solution, as a change of referential won't, fortunately, alter the laws of (classical) mechanics.

In short, the solution presented here was executed with a mathematician/computer scientist mindset (both being professionally conditioned to systematically handle special cases), while the present remark is written with a better understanding of physicists' approach.

You may want to refer to the next exercise, L05E03 for a more orthodox approach.

The existence of a (potential) circular orbit is determined an additional constraint tying the equation of $x(t)$ and $y(t)$. Namely, the equation of motion will describe a circle of radius r, centered on point (a, b), with $(a, b, r) \in \mathbb{R}^3$ if:

$$
(\forall t \geq 0), \ (x(t) - a)^2 + (y(t) - b)^2 = r^2
\tag{11}
$$

Before developing this constraint, we will simplify the expression of our equation of motion. First, let us recall the following trigonometric identity:

$$
\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi
$$

Then, let's introduce two angles φ_x and φ_y such as:

$$
\begin{aligned}
\sin\varphi_x = \alpha_x \quad & \sin\varphi_y = \alpha_y \\
\cos\varphi_x = \beta_x \quad & \cos\varphi_y = \beta_y
\end{aligned}
$$

From, there, we can use the previous identity to rewrite our equations of motions (7) as:

$$
\begin{aligned}
x(t) &= \sin\varphi_x\cos(\omega t - \theta_x) + \cos\varphi_x\sin(\omega t - \theta_x) &&= \sin(\omega t + \varphi_x - \theta_x) = \sin\Omega_x \\
y(t) &= \sin\varphi_y\cos(\omega t - \theta_y) + \cos\varphi_y\sin(\omega t - \theta_y) &&= \sin(\omega t + \varphi_y - \theta_y) = \sin\Omega_y
\end{aligned}
\tag{12}
$$

Obviously with $\Omega_x = \Omega_x(t) = \omega t + \varphi_x - \theta_x$ and $\Omega_y = \Omega_y(t) = \omega t + \varphi_y - \theta_y$.

Let us now develop (11) with those two versions of $x(t)$ and $y(t)$:

$$
\begin{aligned}
(x(t) - a)^2 + (y(t) - b)^2 && = r^2 \\
\Leftrightarrow (\sin \Omega_x - a)^2 + (\sin \Omega_y - b)^2 && = r^2 \\
\Leftrightarrow \sin^2 \Omega_x + \sin^2 \Omega_y - 2(a \sin \Omega_x + b \sin \Omega_y)) + a^2 + b^2 && = r^2
\end{aligned}
$$

For simplicity, we can assume that the circular orbits, if any, will be centered on $(a, b) = (0, 0)$; after all, the choice of the origin is purely conventional, and the law of physics shouldn't change depending on where we decide to place our origin. Which gives:

$$
\sin^2 \Omega_x + \sin^2 \Omega_y = r^2
$$

Which we can rewrite a little bit using the fact that $\sin \varphi = \cos(\varphi - \pi/2)$:

$$
\sin^2 \Omega_x + \cos^2(\Omega_y - \frac{\pi}{2}) = r^2
$$

But we know the Pythagorean identity $\sin^2 \varphi + \cos^2 \varphi = 1$, hence we know there will be circular orbits when:

$$
\begin{cases} r & = 1 \\ \Omega_x & = \Omega_y - \frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r & = 1 \\ \omega t + \varphi_x - \theta_x & = \omega t + \varphi_y - \theta_y - \frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r & = 1 \\ \varphi_x - \theta_x & = \varphi_y - \theta_y - \frac{\pi}{2} \end{cases}
$$

Hence we can see that the only condition relating $x(t)$ and $y(t)$ is that a *phase shift* condition:

$$
\boxed{\varphi_x - \theta_x = \varphi_y - \theta_y - \frac{\pi}{2}}
$$

For all the solutions satisfying that phase-shifts, the period T that we can observed from (12) will be the same:

$$
\boxed{T = \frac{2\pi}{\omega}}
$$

Remark 2. *For a wave function $z(t) = \sin(\omega t + \varphi)$, by definition, $2\pi/\omega$ is the period, and φ the phase shift.*

Remark 3. *The phase shift condition could be rewritten in terms of $\alpha_{x,y}$ and $\beta_{x,y}$.*

Energy conservation
Earlier in the lecture, the kinetic energy has been defined to be *the sum of all the kinetic energies for each coordinate*:

$$
T = \frac{1}{2} \sum_i m_i \dot{x_i}^2 \tag{13}
$$

Which gives us for this system, expliciting the time-dependencies:

$$
T(t) = \frac{1}{2} m \dot{x}(t)^2 + \frac{1}{2} m \dot{y}(t)^2 = \frac{1}{2} m (\dot{x}(t)^2 + \dot{y}(t)^2) \tag{14}
$$

From which we can compute the variation of kinetic energy over time, again using the chain rule:

$$
\begin{aligned}
\frac{d}{dt} T(t) &= \frac{1}{2} m (2\dot{x}(t)\ddot{x}(t) + 2\dot{y}(t)\ddot{y}(t)) \\
&= m(\dot{x}\ddot{x} + \dot{y}\ddot{y})
\end{aligned} \tag{15}
$$

On the other hand, we can compute the variation of potential energy over time from (1), again using the chain rule:

$$\frac{d}{dt}V(t) = \frac{1}{2}k(\frac{d}{dt}x(t)^2 + \frac{d}{dt}y(t)^2)$$
$$= \frac{1}{2}k(2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \qquad (16)$$
$$= k(x(t)\dot{x}(t) + y(t)\dot{y}(t))$$

From (6), we have:

$$x(t) = -\frac{m}{k}\ddot{x}(t)$$
$$y(t) = -\frac{m}{k}\ddot{y}(t) \qquad (17)$$

Injecting in (16):

$$\frac{d}{dt}V(t) = -m(\dot{x}(t)\ddot{x}(t) + \dot{y}(t)\ddot{y}(t))$$
$$= -m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) \qquad (18)$$

Thus from (15) and (18):

$$\frac{d}{dt}E(t) = \frac{d}{dt}T(t) + \frac{d}{dt}V(t) = 0 \quad \square \qquad (19)$$

That is, the total energy E doesn't change over time: it is indeed *conserved*.