The Theoretical Minimum
Classical Mechanics - Solutions
L07E02

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Explain this conservation.*

Let us recall that the referred conserved quantity is:

$$bp_1 + ap_2$$

In the context of the following Lagrangian:

$$L = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - V(aq_1 - bq_2) \tag{1}$$

Because the question is a unclear, we'll make the conservation explicit mathematically, and we'll try to understand the physical meaning of such a quantity being conserved.

As for the previous exercise, we can start by recalling Euler-Lagrange's equations, for instance taken from Equation (13) of the previous chapter ("Lecture 6: The Principle of Least Action"):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_i}}L\right) = \frac{\partial}{\partial q_i}L$$

Which was in the book followed by the definition of the conjugate momentum p_i:

$$p_i = \frac{\partial}{\partial \dot{q_1}}L$$

For our Lagrangian (1), we have for the first half of Euler-Lagrange equations:

$$p_1 \equiv \frac{\partial}{\partial \dot{q_1}}L = \dot{q_1} \qquad p_2 \equiv \frac{\partial}{\partial \dot{q_2}}L = \dot{q_2} \tag{2}$$

$$\frac{d}{dt}p_1 = \dot{p_1} = \ddot{q_1} \qquad \frac{d}{dt}p_2 = \dot{p_2} = \ddot{q_2} \tag{3}$$

Using the chain rule[1] for the other half, with $\varphi(q_i) = aq_1 - bq_2$, we get:

$$\begin{aligned}\frac{\partial}{\partial q_1}L &= -\frac{\partial}{\partial q_1}V(\varphi(q_1)) \\ &= -\frac{\partial}{\partial q_1}\varphi(q_1)\frac{\partial}{\partial q_1}V(\varphi(q_1)) \\ &= -(a\frac{\partial}{\partial q_1}V)(aq_1 - bq_2)\end{aligned} \qquad \begin{aligned}\frac{\partial}{\partial q_2}L &= -\frac{\partial}{\partial q_2}V(\varphi(q_2)) \\ &= -\frac{\partial}{\partial q_2}\varphi(q_2)\frac{\partial}{\partial q_2}V(\varphi(q_2)) \\ &= +(b\frac{\partial}{\partial q_2}V)(aq_1 - bq_2)\end{aligned} \tag{4}$$

[1] `https://en.wikipedia.org/wiki/Chain_rule`

As for the previous exercise, it seems that there a tacit assumption of a symmetry within the potential V so that we can write $V' = \frac{\partial}{\partial q_i} V$; then, combining (2), (3) and (4):

$$\dot{p_1} = -aV'(aq_1 - bq_2) \qquad\qquad \dot{p_2} = +bV'(aq_1 - bq_2)$$

As suggested, let's multiply the first equation by b, the second by a, and sum the result:

$$b\dot{p_1} + a\dot{p_2} = -baV'(aq_1 - bq_2) + abV'(aq_1 - bq_2) = 0$$

By linearity of the derivation, this is equivalent to say that:

$$\frac{d}{dt}(bp_1(t) + ap_2(t)) = 0$$

Which indeed means that $bp_1(t) + ap_2(t) \in \mathbb{R}$ is a indeed a constant over time, i.e. that it is conserved (over time).

Now, let's see if we can understand what this means physically: essentially, $aq_1 - bq_2$ means that we're scaling the "position" of the particles respectively by a and b, and make the potential depends on the resulting distance.

The conserved quantity is the "conjugate" of this distance