The Theoretical Minimum

Classical Mechanics - Solutions

L07E04

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Show this to be true.*

Where "this" refers to the fact that this Lagrangian (Equation (8)):

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \tag{1}$$

does not change to first order in δ, for the infinitesimal transformation described by e.g. Equations (12):

$$\begin{aligned} \delta_v x &= y\delta \\ \delta_v y &= -x\delta \end{aligned} \tag{2}$$

The transformation to the derivatives over time of x and y has already been established in Equations (11):

$$\begin{aligned} \dot{x} &\to \dot{x} + \dot{y}\delta \\ \dot{y} &\to \dot{y} - \dot{x}\delta \end{aligned} \tag{3}$$

Let's then perform the substitution described by (2) and (3) in the Lagrangian (1):

$$\begin{aligned} L &= \frac{m}{2}((\dot{x} + \dot{y}\delta)^2 + (\dot{y} - \dot{x}\delta)^2) - V((x + y\delta)^2 + (y - x\delta)^2) \\ &= \frac{m}{2}\left(\left(\dot{x}^2 + 2\dot{x}\dot{y}\delta + (\dot{y}\delta)^2\right) + \left(\dot{y}^2 - 2\dot{y}\dot{x}\delta + (\dot{x}\delta)^2\right)\right) \\ &\quad - V\left(\left(x^2 + 2xy\delta + (y\delta)^2\right) + \left(y^2 - 2yx\delta + (x\delta)^2\right)\right) \\ &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \delta^2(\dot{x}^2 + \dot{y}^2)\right) - V\left(x^2 + y^2 + \delta^2(x^2 + y^2)\right) \end{aligned}$$

Now, as we care about first-order changes in δ only, changes proportional to $\delta^n|_{n\geq 2}$ will be negligible; it follows that the Lagrangian is indeed unchanged:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \quad \square$$