The Theoretical Minimum

Classical Mechanics - Solutions

L08E01

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Start with the Lagrangian* $\frac{m\dot{x}^2}{2} - \frac{k}{2}x^2$ *and show that if you make the change in variables* $q = (km)^{1/4}x$*, the Lagrangian has the form of Eq.* (14)*. What is the connection among* k*,* m *and* ω *?*

Let's recall some context. We're in the case of a the harmonic oscillator, covered in-depth in L03E04.

More precisely, the authors just shown us how the harmonic oscillator can be used as an approximation in the case of an equilibrium state that is slightly disturbed. Physically, you can for instance consider the case of a pendulum: if the oscillations are kept small, then the mass of the pendulum will describe something that *locally* looks like the bottom of a quadratic polynomial (e.g. $ax^2 + bx + c = 0$). This is similar to how the derivative is a local linear approximation, except things jiggle a little more, so to speak.

Eq. (14) of the book refers to the following Lagrangian:

$$L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$$

Now let's simply perform the required change of variable. Both m and k are constants, so $\dot{q} = (km)^{1/4}\dot{x}$, and:

$$L = \frac{1}{2\omega}\sqrt{km}\dot{x}^2 - \frac{\omega}{2}\sqrt{km}x^2$$

Finally, if we can try to identify the previous expression with the one we're supposed to find, by finding a relation between ω, k and m. We have respectively for each term:

$$m = \frac{\sqrt{km}}{\omega}; \qquad k = \omega\sqrt{km}$$

$$\Leftrightarrow \omega = \frac{\sqrt{k}\sqrt{m}}{\sqrt{m}^2}; \qquad \omega = \frac{\sqrt{k}^2}{\sqrt{k}\sqrt{m}}$$

$$\Leftrightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

So we can consistently identify both expression by defining ω as previously stated. Such an ω furthermore matches the usual definition we have in a harmonic oscillator setting.