The Theoretical Minimum

Classical Mechanics - Solutions

L11E01

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Confirm Eq.* (3)*. Also prove that*

$$V_i A_j - V_j A_i = \sum_k \epsilon_{ijk} \left(\vec{V} \times \vec{A}\right)_i$$

Let's recall that the Levi-Civita symbol is defined by:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

Eq. (3) refers to:

$$\left(\vec{V} \times \vec{A}\right)_i = \sum_j \sum_k \epsilon_{ijk} V_j A_k$$

So the idea is to express the components of the cross-product of two 3D vectors with the Levi-Civita symbol. Let's have a look at the components of the cross-product of two vectors:

$$\begin{aligned} \left(\vec{V} \times \vec{A}\right)_x &= V_y A_z - V_z A_y \\ \left(\vec{V} \times \vec{A}\right)_y &= V_z A_x - V_x A_z \\ \left(\vec{V} \times \vec{A}\right)_z &= V_x A_y - V_y A_x \end{aligned}$$

Remark 1. *There's a typo in the book the last term should contain an A_x, but the book says its an A_z. There's another typo in the exercise actually; we'll get to it in a moment.*

Observe that somehow, all those components are "equivalent", or "symmetric": for instance, we can get the second line from the first, by renaming in the first x by y, y by z and z by x.

This implies that to verify Eq. (3), we can satisfy ourselves with doing it only for one component, as the procedure would be exactly similar for the two others. So, let's get going, for instance by trying to prove the first line:

$$
\begin{aligned}
\left(\vec{V}\times\vec{A}\right)_x &= \sum_j\sum_k \epsilon_{xjk} V_j A_k \\
&= \sum_k \underbrace{\epsilon_{xxk}}_{=0} V_x A_k + \sum_k \epsilon_{xyk} V_y A_k + \sum_k \epsilon_{xzk} V_z A_k \\
&= \sum_k \Big(\epsilon_{xyk} V_y A_k + \epsilon_{xzk} V_z A_k \Big) \\
&= \underbrace{\epsilon_{xyx}}_{=0} V_y A_x + \underbrace{\epsilon_{xzx}}_{=0} V_z A_x + \underbrace{\epsilon_{xyy}}_{=0} V_y A_y + \epsilon_{xzy} V_z A_y + \epsilon_{xyz} V_y A_z + \underbrace{\epsilon_{xzz}}_{=0} V_z A_z \\
&= \underbrace{\epsilon_{xzy}}_{=-1} V_z A_y + \underbrace{\epsilon_{xyz}}_{=1} V_y A_z \\
&= V_y A_z - V_z A_y \quad \square
\end{aligned}
$$

For similar reasons (symmetry), we only have to consider the case e.g. $i = x$ of the remaining equation to be done with it, as the two others would be derived identically, but for some systematic renaming.

We then have three sub-cases, depending on the value of j. If $j = i(= x)$, then one one side:

$$V_i A_j - V_j A_i = V_i A_i - V_i A_i = 0$$

And on the other:

$$\sum_k \underbrace{\epsilon_{iik}}_{=0} \left(\vec{V}\times\vec{A}\right)_i = 0$$

And so the equation holds. Now let's consider the case where $j = y$. One one side we have:

$$V_i A_j - V_j A_i = V_x A_y - V_y A_x$$

And on the other:

$$
\begin{aligned}
\sum_k \epsilon_{xjk} \left(\vec{V}\times\vec{A}\right)_x &= \sum_k \epsilon_{xjk} \Big(V_y A_z - V_z A_y \Big) \\
&= \sum_k \epsilon_{xyk} \Big(V_y A_z - V_z A_y \Big) \\
&= \Big(V_y A_z - V_z A_y \Big) (\underbrace{\epsilon_{xyx}}_{=0} + \underbrace{\epsilon_{xyy}}_{=0} + \underbrace{\epsilon_{xyz}}_{=1}) \\
&= V_y A_z - V_z A_y
\end{aligned}
$$

Well, the computations are right, but obviously the result isn't! There's a typo in the book: we're expected to prove:

$$V_i A_j - V_j A_i = \sum_k \epsilon_{ijk} \left(\vec{V}\times\vec{A}\right)_k$$

So, let's start again the development of the right hand side:

$$
\begin{aligned}
\sum_k \epsilon_{xjk} \left(\vec{V}\times\vec{A}\right)_k &= \sum_k \epsilon_{xyk} \left(\vec{V}\times\vec{A}\right) \\
&= \underbrace{\epsilon_{xyx}}_{=0} \left(\vec{V}\times\vec{A}\right)_x + \underbrace{\epsilon_{xyy}}_{=0} \left(\vec{V}\times\vec{A}\right)_y + \underbrace{\epsilon_{xyz}}_{=1} \left(\vec{V}\times\vec{A}\right)_z \\
&= \left(\vec{V}\times\vec{A}\right)_z \\
&= V_x A_y - V_y A_x \quad \square
\end{aligned}
$$

Which corresponds to the left-hand side. Let's do it once more with $j = z$. On one side:

$$V_i A_j - V_j A_i = V_x A_z - V_z A_x$$

On the other:

$$
\begin{aligned}
\sum_k \epsilon_{xjk}\left(\vec{V}\times\vec{A}\right)_k &= \sum_k \epsilon_{xzk}\left(\vec{V}\times\vec{A}\right) \\
&= \underbrace{\epsilon_{xzx}}_{=0}\left(\vec{V}\times\vec{A}\right)_x + \underbrace{\epsilon_{xzy}}_{=-1}\left(\vec{V}\times\vec{A}\right)_y + \underbrace{\epsilon_{xzz}}_{=0}\left(\vec{V}\times\vec{A}\right)_z \\
&= -\left(\vec{V}\times\vec{A}\right)_y \\
&= -(V_z A_x - V_x A_z) \\
&= V_x A_z - V_z A_x \quad \square
\end{aligned}
$$