

# The Theoretical Minimum

## Classical Mechanics - Solutions

L11E04

Last version: [tales.mbivert.com/on-the-theoretical-minimum-solutions/](https://tales.mbivert.com/on-the-theoretical-minimum-solutions/) or [github.com/mbivert/ttm](https://github.com/mbivert/ttm)

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**Exercise 1.** *Using the Hamiltonian, Eq. (24), work out Hamilton's equations of motion and show that you just get back to the Newton-Lorentz equation of motion.*

The Hamiltonian is:

$$H = \frac{1}{2m} \sum_i \left( p_i - \frac{e}{c} A_i(\mathbf{q}) \right)^2$$

Hamilton's equation of motion are given by the pair:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}; \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Let's get started with the first one:

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ &= \frac{1}{2m} \frac{\partial}{\partial p_j} \sum_i \left( p_i - \frac{e}{c} A_i(\mathbf{q}) \right)^2 \\ &= \frac{1}{2m} \frac{\partial}{\partial p_j} \sum_i \left( p_i^2 - 2\frac{e}{c} p_i A_i(\mathbf{q}) + \left(\frac{e}{c}\right)^2 A_i(\mathbf{q})^2 \right) \\ &= \frac{1}{2m} \sum_i \left( \frac{\partial p_i^2}{\partial p_j} - 2\frac{e}{c} \left( \frac{\partial p_i}{\partial p_j} A_i(\mathbf{q}) + p_i \underbrace{\frac{\partial A_i(\mathbf{q})}{\partial p_j}}_{=0} \right) + \left(\frac{e}{c}\right)^2 \underbrace{\frac{\partial A_i(\mathbf{q})^2}{\partial p_j}}_{=0} \right) \\ &= \frac{1}{2m} \sum_i \left( \frac{\partial p_i^2}{\partial p_j} - 2\frac{e}{c} \frac{\partial p_i}{\partial p_j} A_i(\mathbf{q}) \right) \\ &= \frac{1}{2m} \left( 2p_j - 2\frac{e}{c} A_j(\mathbf{q}) \right) \\ &= \boxed{\frac{1}{m} \left( p_j - \frac{e}{c} A_j(\mathbf{q}) \right)} \end{aligned}$$

We've found back the expression of the moment, tweaked a little. Let's now take the time derivative of the previous equation to get:

$$\begin{aligned} \ddot{q}_j &= \frac{1}{m} \left( \dot{p}_j - \frac{e}{c} \dot{A}_j(\mathbf{q}) \right) \\ \Leftrightarrow m\ddot{q}_j &= \dot{p}_j - \frac{e}{c} \dot{A}_j(\mathbf{q}) \end{aligned}$$

Which starts to look like an equation of motion. We'll develop  $\dot{A}_j(\mathbf{q})$  using the chain rule later. For now, let's compute  $\dot{p}_j$ , but this time, let's make our life a bit simpler by using the chain rule:

$$\begin{aligned}
\dot{p}_j &= -\frac{\partial H}{\partial q_j} \\
&= -\frac{1}{2m} \frac{\partial}{\partial q_j} \sum_i \underbrace{\left(p_i - \frac{e}{c} A_i(\mathbf{q})\right)^2}_{=\phi} \\
&= -\frac{1}{2m} \sum_i 2\phi \frac{\partial \phi}{\partial q_j} \\
&= \sum_i \frac{1}{m} \underbrace{\left(p_i - \frac{e}{c} A_i(\mathbf{q})\right)}_{=\dot{q}_i} \frac{e}{c} \frac{\partial A_i(\mathbf{q})}{\partial q_j} \\
&= \boxed{\frac{e}{c} \sum_i \dot{q}_i \frac{\partial A_i(\mathbf{q})}{\partial q_j}}
\end{aligned}$$

Now let's use the multi-dimensional chain rule to develop  $\dot{A}_j(\mathbf{q})$ :

$$\dot{A}_j(\mathbf{q}) = \dot{A}_j(q_x(t), q_y(t), q_z(t)) = \sum_i \frac{\partial A_j(\mathbf{q})}{\partial q_i} \dot{q}_i$$

Finally, let's use this and the expression of  $\dot{p}_j$  to rewrite our embryo of motion equation:

$$\begin{aligned}
m\ddot{q}_j &= \dot{p}_j - \frac{e}{c} \dot{A}_j(\mathbf{q}) \\
&= \sum_i \dot{q}_i \frac{e}{c} \frac{\partial A_i(\mathbf{q})}{\partial q_j} - \frac{e}{c} \sum_i \frac{\partial A_j(\mathbf{q})}{\partial q_i} \dot{q}_i \\
&= \frac{e}{c} \sum_i \dot{q}_i \left( \frac{\partial A_i(\mathbf{q})}{\partial q_j} - \frac{\partial A_j(\mathbf{q})}{\partial q_i} \right) \\
&= \frac{e}{c} \left( \dot{q}_j \underbrace{\left( \frac{\partial A_j(\mathbf{q})}{\partial q_j} - \frac{\partial A_j(\mathbf{q})}{\partial q_j} \right)}_{=0} + \dot{q}_k \underbrace{\left( \frac{\partial A_k(\mathbf{q})}{\partial q_j} - \frac{\partial A_j(\mathbf{q})}{\partial q_k} \right)}_{=B_l} + \dot{q}_l \underbrace{\left( \frac{\partial A_l(\mathbf{q})}{\partial q_j} - \frac{\partial A_j(\mathbf{q})}{\partial q_l} \right)}_{=-B_k} \right) \\
&= \frac{e}{c} (B_l \dot{q}_k - B_k \dot{q}_l)
\end{aligned}$$

And so for  $j, k, l$  three distinct elements of  $\{x, y, z\}$ , we indeed have found our equations of Newton-Lorentz:

$$\boxed{m\ddot{q}_j = \frac{e}{c} (B_l \dot{q}_k - B_k \dot{q}_l)}$$