The Theoretical Minimum

Quantum Mechanics - Solutions

L01E02

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Show that the inner product defined by Eq.* 1.2 *satisfies all the axioms of inner products.*

Let us recall the two relevant axioms:

Axiom 1.

$$\langle C|\Big(|A\rangle + |B\rangle\Big) = \langle C|A\rangle + \langle C|B\rangle$$

Axiom 2.

$$\langle B|A\rangle = \langle A|B\rangle^*$$

Where z^* is the complex conjugate of $z \in \mathbb{C}$

And let us recall Eq. 1.2 of the book:

$$\begin{aligned}\langle B|A\rangle &= \begin{pmatrix}\beta_1^* & \beta_2^* & \beta_3^* & \beta_4^* & \beta_5^*\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix}\\ &= \beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \beta_3^*\alpha_3 + \beta_4^*\alpha_4 + \beta_5^*\alpha_5\end{aligned}$$

For the first axiom, considering $\langle C| = (\gamma_1^* \ \gamma_2^* \ \gamma_3^* \ \gamma_4^* \ \gamma_5^*)$:

$$\begin{aligned}\langle C|\Big(|A\rangle + |B\rangle\Big) &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3+\beta_3\\ \alpha_4+\beta_4\\ \alpha_5+\beta_5\end{pmatrix}\\ &= \gamma_1^*(\alpha_1+\beta_1) + \gamma_2^*(\alpha_2+\beta_2) + \gamma_3^*(\alpha_3+\beta_3) + \gamma_4^*(\alpha_4+\beta_4) + \gamma_5^*(\alpha_5+\beta_5)\\ &= \big(\gamma_1^*\alpha_1 + \gamma_2^*\alpha_2 + \gamma_3^*\alpha_3 + \gamma_4^*\alpha_4 + \gamma_5^*\alpha_5\big) + \big(\gamma_1^*\beta_1 + \gamma_2^*\beta_2 + \gamma_3^*\beta_3 + \gamma_4^*\beta_4 + \gamma_5^*\beta_5\big)\\ &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix} + \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^*\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}\\ &= \langle C|A\rangle + \langle C|B\rangle \quad \square\end{aligned}$$

Before checking the second axiom, let us observe that for $(a, b) = (x_a + iy_a, x_b + iy_b) \in \mathbb{C}^2$:

$$\begin{aligned}
(ab)^* &= \Big((x_a + iy_a) \times (x_b + iy_b)\Big)^* \\
&= \Big(x_a x_b - y_a y_b + i(x_b y_a + x_a y_b)\Big)^* \\
&= x_a x_b - y_a y_b - i(x_b y_a + x_a y_b) \\
&= (x_a - iy_a) \times (x_b - iy_b) \\
&= a^* b^*
\end{aligned}$$

Remark 1. *We could have derived it using complex numbers' exponential's form:*

$$\begin{aligned}
(ab)^* &= \Big(r_a r_b e^{i(\theta_a + \theta_b)}\Big)^* \\
&= r_a r_b e^{-i(\theta_a + \theta_b)} \\
&= a^* b^*
\end{aligned}$$

Hence, regarding the second axiom:

$$\begin{aligned}
\langle B|A\rangle &= \Big(\big(\,\langle B|A\rangle\,\big)^*\Big)^* \\
&= \Big(\big(\beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \beta_3^*\alpha_3 + \beta_4^*\alpha_4 + \beta_5^*\alpha_5\big)^*\Big)^* \\
&= \Big(\beta_1\alpha_1^* + \beta_2\alpha_2^* + \beta_3\alpha_3^* + \beta_4\alpha_4^* + \beta_5\alpha_5^*\Big)* \\
&= \Big(\alpha_1^*\beta_1 + \alpha_2^*\beta_2 + \alpha_3^*\beta_3 + \alpha_4^*\beta_4 + \alpha_5^*\beta_5\Big)* \\
&= \left(\begin{pmatrix}\alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* & \alpha_5^*\end{pmatrix}\begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5\end{pmatrix}\right)^* \\
&= \langle A|B\rangle^* \quad \square
\end{aligned}$$