The Theoretical Minimum

Quantum Mechanics - Solutions

L03E02

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Prove that Eq.* 3.16 *is the unique solution to Eqs.* 3.14 *and* 3.15*.*

Let's recall all the equations, 3.14, 3.15 and 3.16

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

By developing the matrix product and identifying the vectors components, the first two equations make a system of four equations involving four unknowns $(\sigma_z)_{11}$, $(\sigma_z)_{12}$, $(\sigma_z)_{21}$ and $(\sigma_z)_{22}$:

$$\begin{cases} 1(\sigma_z)_{11} + 0(\sigma_z)_{12} &= 1 \\ 1(\sigma_z)_{21} + 0(\sigma_z)_{22} &= 0 \\ 0(\sigma_z)_{11} + 1(\sigma_z)_{12} &= 0 \\ 0(\sigma_z)_{21} + 1(\sigma_z)_{22} &= -1 \end{cases} \Leftrightarrow \begin{cases} (\sigma_z)_{11} &= 1 \\ (\sigma_z)_{21} &= 0 \\ (\sigma_z)_{12} &= 0 \\ (\sigma_z)_{22} &= -1 \end{cases} \Leftrightarrow \boxed{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \quad \square \tag{4}$$

Remark 1. *Observe that we are (were) trying to build a Hermitian operator with eigenvalues* $+1$ *and* -1*. The fundamental theorem / real spectral theorem, assures us that Hermitian operators are diagonalizable, hence there exists a basis in which the operator can be represented by a* 2×2 *matrix containing the eigenvalues on its diagonal:*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which is exactly the matrix we've found.

But now of course, you'd be wondering: wait a minute, right after this exercise, we're trying to build σ_x*, which also has those same eigenvalues* $+1$ *and* -1*, what's the catch?*

Well, remember the diagonalization process: M *diagonalizable means that there's a basis where it's diagonal. That is, there's a change of basis, which is an invertible linear function, which has a matrix representation* P*, such that the linear operation represented by* M *in a starting basis is now represented by a diagonal matrix* D*:*

$$M = PDP^{-1}$$

Furthermore:

- *The elements on the diagonal of* D *are the eigenvalues;*
- *The columns of* P *are the corresponding eigenvectors*

So regarding σ_x, we still have a

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

But the catch is that before for σ_z, P was the identity matrix I_2 (because of our choice for $|u\rangle$ and $|d\rangle$). But now, given our values for $|r\rangle$ and $|l\rangle$, we have:

$$|r\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \textit{and} \quad |l\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \Rightarrow \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note that the column order matters: the first column of P must be $|r\rangle$, and the first column of D must contain the eigenvalue associated to $|r\rangle$. But:

$$\sigma_x = PDP^{-1} \Leftrightarrow \sigma_x P = PD(\underbrace{P^{-1}P}_{:=I_2}) = PD = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Hence,

$$\sigma_x P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} (\sigma_x)_{11} & (\sigma_x)_{12} \\ (\sigma_x)_{21} & (\sigma_x)_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Solving for the components of σ_x:

$$\Leftrightarrow \begin{cases} (\sigma_x)_{11} + (\sigma_x)_{12} = 1 \\ (\sigma_x)_{11} - (\sigma_x)_{12} = -1 \\ (\sigma_x)_{21} + (\sigma_x)_{22} = 1 \\ (\sigma_x)_{21} - (\sigma_x)_{22} = 1 \end{cases}$$

Which indeed yields the expected Pauli matrix, as described in the book, and computed by the authors using a different approach:

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

And obviously, the same can be done for σ_y : that's to say that, reassuringly, we reach the same results using pure linear algebra.