The Theoretical Minimum

Quantum Mechanics - Solutions

L03E04

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Let $n_z = \cos\theta$, $n_x = \sin\theta\cos\phi$ and $n_y = \sin\theta\sin\phi$. Angles θ and ϕ are defined according to the usual conventions for spherical coordinates (Fig. 3.2). Compute the eigenvalues and eigenvectors for the matrix of Eq. 3.23.*

Let's recall Eq. 3.23, which is general form of the spin 3-vector operator:

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix} = \begin{pmatrix} \cos\theta & (\sin\theta\cos\phi - i(\sin\theta\sin\phi)) \\ (\sin\theta\cos\phi + i(\sin\theta\sin\phi)) & -\cos\theta \end{pmatrix}$$

Observe (e.g. from the trigonometric circle) that:

$$\cos\theta = \cos(-\theta); \qquad \sin\theta = -\sin(-\theta)$$

Hence:

$$\exp(-i\theta) := \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

And we can simplify our previous expression of σ_n to:

$$\sigma_n = \begin{pmatrix} \cos\theta & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta \end{pmatrix}$$

Note that as we're now in the general case, we indeed have two degrees of freedom, encoded by the two angles θ and ϕ; the *why* has been explicited in subsection 2.5.

We're still confronted to a spin operator: we expect the eigenvalues to be $+1$ and -1[1]. But let's check this first: an eigenvector $|\lambda\rangle$ associated to an eigenvalue λ must obey:

$$\sigma_n|\lambda\rangle = \lambda|\lambda\rangle$$

$$\Leftrightarrow \sigma_n|\lambda\rangle - \lambda|\lambda\rangle = 0 \Leftrightarrow (\sigma_n - I_2\lambda)|\lambda\rangle = 0$$

But eigenvectors are non-zero, hence, again with 0_2 being the 2×2 zero matrix:

$$\Leftrightarrow \sigma_n - I_2\lambda = 0_2$$

[1] Remember from the real spectral theorem, or as the authors call it, the *fundamental theorem*, that because we have a Hermitian matrix, we know it's diagonalizable, that its eigenvalues are real, and that the corresponding eigenvectors form a orthogonal basis

And so this matrix $\sigma_n - I_2\lambda$ cannot be invertible[2]. This translates to a condition on the determinant:

$$\begin{aligned}
\det(\sigma_n - I_2\lambda) = 0 \quad &\Leftrightarrow \quad \begin{vmatrix} \cos\theta - \lambda & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0 \\
&\Leftrightarrow \quad -(\cos\theta - \lambda)(\cos\theta + \lambda) - \underbrace{\exp(i\phi)\exp(-i\phi)}_{=1}\sin^2\theta = 0 \\
&\Leftrightarrow \quad -(\cos^2\theta - \lambda^2) - \sin^2\theta = 0 \\
&\Leftrightarrow \quad \lambda^2 - (\underbrace{\sin^2\theta + \cos^2\theta}_{=1}) = 0 \\
&\Leftrightarrow \quad \lambda^2 = 1 \\
&\Leftrightarrow \quad \boxed{\lambda = \begin{cases} +1 \\ -1 \end{cases}}
\end{aligned}$$

The remaining difficulty is then in finding the eigenvectors. We can use the following argument[3].

Consider as a first guess an eigenvector of the form:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \qquad (z_1, z_2) \in \mathbb{C}^2$$

We can put both complex numbers in exponential form:

$$\begin{pmatrix} r_1\exp(i\phi_1) \\ r_2\exp(i\phi_2 \end{pmatrix} = \exp(i\phi_1)\begin{pmatrix} r_1 \\ r_2\exp(i(\phi_2 - \phi_1)) \end{pmatrix}; \qquad (r_1, r_2, \phi_1, \phi_2) \in \mathbb{R}^4$$

We can then ignore the general phase factor $\exp(i\phi_1)$, e.g. set $\phi_1 = 0$. Furthermore, we want the vector to be normalized (this is an eigenvector associated to the eigenvalue of a Hermitian operator: it must be normalized per the real spectral theorem), i.e.

$$|r_1|^2 + |r_2\exp(i\phi_2)|^2 = 1 \Leftrightarrow |r_1|^2 + |r_2|^2 = 1$$

But we're then losing a degree of freedom, meaning, r_1 and r_2 are not independent from each other: we can express them both in term of a single parameter, as long as the previous equation is satisfied. We can choose, as it'll make computation easier, $r_1 = \cos\alpha$, $r_2 = \sin\alpha$, with $\alpha \in \mathbb{R}$. Finally, let's rename $\phi_2 = \phi_\alpha$[4], which brings us to consider eigenvectors of the form:

$$\begin{pmatrix} \cos\alpha \\ \exp(i\phi_\alpha)\sin\alpha \end{pmatrix}$$

As for the previous exercise, we can use two different parameter α and β for each eigenvector. Again, because of the diagonalization process, we have the following relation

$$\sigma_n = PDP^{-1} \Leftrightarrow \sigma_n P = PD(\underbrace{PP^{-1}}_{:=I_2}) = PD = P\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

But the columns of P must contain our eigenvectors, so this is equivalent to:

$$\underbrace{\begin{pmatrix} \cos\theta & \exp(-i\phi)\sin\theta \\ \exp(i\phi)\sin\theta & -\cos\theta \end{pmatrix}}_{=\sigma_n}\underbrace{\begin{pmatrix} \cos\alpha & \cos\beta \\ \exp(i\phi_\alpha)\sin\alpha & \exp(i\phi_\beta)\sin\beta \end{pmatrix}}_{=P} = \begin{pmatrix} \cos\alpha & \cos\beta \\ \exp(i\phi_\alpha)\sin\alpha & \exp(i\phi_\beta)\sin\beta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\cos\beta \\ \exp(i\phi_\alpha)\cos\alpha & -\exp(i\phi_\beta)\sin\beta \end{pmatrix}$$

[2]Again for otherwise, as recalled in L03E03, multiply both sides of the equation by its inverse, get an identity on the left-hand-side and still the zero matrix on the right-hand-side

[3]`https://physics.stackexchange.com/a/720025`

[4]Note that I'm not yet identifying ϕ_α with ϕ; this will come naturally later on

Let's perform the matrix multiplication on the left:

$$\begin{pmatrix} \cos\theta\cos\alpha + \exp(i(\phi_\alpha - \phi))\sin\theta\sin\alpha & \cos\theta\cos\beta + \exp(i(\phi_\beta - \phi))\sin\theta\sin\beta \\ \exp(i\phi)\sin\theta\cos\alpha - \exp(i\phi_\alpha)\cos\theta\sin\alpha & \exp(i\phi)\sin\theta\cos\beta - \exp(i\phi_\beta)\cos\theta\sin\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\cos\beta \\ \exp(i\phi_\alpha)\cos\alpha & -\exp(i\phi_\beta)\sin\beta \end{pmatrix}$$

From which we can extract the following system of equations:

$$\begin{cases} \cos\theta\cos\alpha + \exp(i(\phi_\alpha - \phi))\sin\theta\sin\alpha & = \cos\alpha \\ \cos\theta\cos\beta + \exp(i(\phi_\beta - \phi))\sin\theta\sin\beta & = -\cos\beta \end{cases}$$

Remark 1. *As for the previous exercise, I leave it to you to check that the solution we'll find for this system also solve the two other omitted equations.*

It's tempting to set $\phi = \phi_\alpha = \phi_\beta$, but can we do so? Well, we know the two eigenvectors will have to be orthogonal: this adds an additional constraint, which decrease our degrees of freedom by one, meaning there's one superfluous variable in $\{\alpha, \beta, \phi_\alpha, \phi_\beta\}$. We can *choose* to implement this constraint by setting $\phi_\alpha = \phi_\beta$.

From there, we can indeed set $\phi_\alpha = \phi_\beta = \phi$, as this allows us to solve the equation for α and β more easily:

$$\Leftrightarrow \begin{cases} \cos\theta\cos\alpha + \sin\theta\sin\alpha & = \cos\alpha \\ \sin\theta\cos\beta - \cos\theta\sin\beta & = -\cos\beta \end{cases}$$

Which is exactly the same system we had for the previous exercise, which was solved by:

$$\begin{cases} \alpha & = \theta/2 \\ \beta & = \frac{1}{2}(\theta + \pi) \end{cases}$$

With the same trigonometric identities as for the previous exercise:

$$\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha; \qquad \sin(\alpha + \frac{\pi}{2}) = \cos\alpha$$

We reach the following eigenvectors

$$\boxed{\begin{cases} |+1\rangle & = \begin{pmatrix} \cos\alpha \\ \exp(i\phi)\sin\alpha \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \exp(i\phi)\sin(\theta/2) \end{pmatrix} \\ |-1\rangle & = \begin{pmatrix} \cos\beta \\ \exp(i\phi)\sin\beta \end{pmatrix} = \begin{pmatrix} -\sin(\theta/2) \\ \exp(i\phi)\cos(\theta/2) \end{pmatrix} \end{cases}}$$

Alright, let's make the same verifications the authors did in the book after the previous exercise. First, we get the expected eigenvalues $+1$, -1, which are the only two eigenvalues we have for a spin operator.

Then the two eigenvectors must be orthogonal, indeed (I only do it one way; the other is trivially similar):

$$\langle +1|-1\rangle = \begin{pmatrix} \cos(\theta/2) & \exp(-i\phi)\sin(\theta/2) \end{pmatrix} \begin{pmatrix} -\sin(\theta/2) \\ \exp(i\phi)\cos(\theta/2) \end{pmatrix}$$

$$= -\cos(\theta/2)\sin(\theta/2) + \exp(-i\phi + i\phi)\cos(\theta/2)\sin(\theta/2) = 0$$

Finally, if we prepare a spin along the z-axis in the up state $|u\rangle$, then rotate our apparatus to lie along the $\hat{n}$ axis, which *is not* restricted to the xz-plane anymore, we have according to the fourth principle[5]:

$$P(+1) = |\langle u|+1\rangle|^2 = \cos^2(\theta/2)$$

$$P(-1) = |\langle u|-1\rangle|^2 = \sin^2(\theta/2)$$

Which then lead to the exact same computation regarding the expected value for the measurement:

$$\langle \sigma_n \rangle = \sum_i \lambda_i P(\lambda_i) = (+1)\cos^2(\theta/2) + (-1)\sin^2(\theta/2) = \boxed{\cos\theta}$$

Note also that $P(+1) + P(-1) = 1$.

[5] Don't hesitate to get back to the definition of $|u\rangle$ and that of the inner-product if this isn't clear enough.