The Theoretical Minimum

Quantum Mechanics - Solutions

L03E05

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Suppose that a spin is prepared so that $\sigma_m = +1$. The apparatus is then rotated to the $\hat{n}$ direction and σ_n is measured. What is the probability that the result is $+1$? Note that $\sigma_m = \sigma \cdot \hat{m}$, using the same convention we used for σ_n.*

There are essentially two ways of solving the issue.

The first one, and the simplest, is to observe that if we consider $\hat{n}$ in a frame of reference where $\hat{m}$ acts as our $z-axis$, then we're essentially in the case of our previous exercise: we've prepared a spin in the "up" state (now corresponding to a state where $\sigma_m = +1$), we've moved our apparatus away from $\hat{m}$ by a a certain angle θ[1], and we know from the previous exercise that the probability of measuring a $+1$ after aligning our apparatus with the $\hat{n}$ axis is now

$$\boxed{P(+1) = \cos^2 \frac{\theta}{2}}$$

Which is exactly what we wanted to show (the answer is given in the book by the authors, after the exercise).

I'll only draft the second approach, as I expect it to be more time consuming[2]. The idea is not to rely on the previous observation, and to consider that we've prepared to spin so that $\sigma_m = +1$, which means the state of the system is the eigenvector corresponding to this eigenvalue, which we know from the previous exercise, with θ_m the angle between the z-axis and $\hat{m}$, and ϕ_m the angle between the x-axis and the projection of $\hat{m}$ on the xy-plane:

$$|+1_m\rangle = \begin{pmatrix} \cos(\theta_m/2) \\ \exp(i\phi_m)\sin(\theta_m/2) \end{pmatrix}$$

If we then align the apparatus in the $\hat{n}$ direction, with corresponding θ_n / ϕ_n angles, *which are relative to the z-axis, not $\hat{m}$* , we now, by the same result, that the eigenvector corresponding to the probability of measuring a $+1$ in the $\hat{n}$ direction is:

$$|+1_n\rangle = \begin{pmatrix} \cos(\theta_n/2) \\ \exp(i\phi_n)\sin(\theta_n/2) \end{pmatrix}$$

Then, the probability to measure a $+1$ is given, again by using the fourth principle:

$$P(+1) = |\langle +1_m | +1_n \rangle|^2$$

[1] θ really is the angle between $\hat{m}$ and $\hat{n}$, not some angle between $\hat{n}$ and the "real" z-axis

[2] And hopefully, valid...

We would then need to develop the inner-product between the two state vectors, and find a way to identify it with the half-angle between $\hat{n}$ and $\hat{m}$.

All the difficulty is then in expressing this half-angle in terms of our four angles (θ_m, ϕ_m, θ_n, ϕ_n). I *suppose* we get some insightful elements by cleverly:

- Expressing $\hat{m}$ and $\hat{n}$ both in rectangular coordinates;
- Observing that by the regular 3-vector dot product, $\hat{n} \cdot \hat{m} = \|\hat{n}\|\|\hat{m}\| \cos\theta_{mn} = \cos\theta_{mn}$ (where θ_{mn} is the angle between $\hat{m}$ and $\hat{n}$
- Observing that $\cos\frac{\theta_{mn}}{2} = \frac{1}{\sqrt{2}}\hat{n} \cdot (\hat{n} + \hat{m})$ (again from the regular 3-vector dot product, as $\hat{n} + \hat{m}$ will be a (non-unitary) vector bisecting θ_{mn}[3])

[3] `https://math.stackexchange.com/a/2285989`: the parallelogram involved in the sum of two vectors in a rhombus.