The Theoretical Minimum

Quantum Mechanics - Solutions

L04E03

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Go back to the definition of Poisson brackets in Volume I and check that the identification in Eq.* 4.21 *is dimensionally consistent. Show that without the factor* $\hbar$*, it would not be.*

Let's recall first Eq. 4.21, where $[.,.]$ is the commutator and $\{.,.\}$ the Poisson brackets:

$$[F,G] \iff i\hbar\{F,G\}$$

The Poisson brackets are defined in *Volume I*, Eq. (9) at the end of Lecture 9 (The Phase Space Fluid and the Gibbs-Liouville Theorem), as:

$$\{F,G\} := \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

Where the p_i are the generalized momentum, and q_i are the generalized coordinates. Recall that a momentum is typically defined as a mass in motion, while the coordinates are simply distances to an origin:

$$[p_i] = \text{kg.m.s}^{-1}; \qquad [q_i] = \text{m}$$

For clarity, let's rewrite one of those partial derivative in terms of a limit:

$$\frac{\partial F}{\partial q_i} = \lim_{\epsilon\to 0}\frac{F(q_i+\epsilon) - F(q_i)}{\epsilon}$$

First ϵ must be of the same dimension than q_i is this case, for otherwise $q_i+\epsilon$ is ill-defined; more generally it'll have the same dimension that the dimension of the differentiation variable.

Second, observe that, again because otherwise we'd be adding carrots and potatoes:

$$\left[\sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right] = \left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right], \quad \text{for any arbitrary } i \text{ that is}$$

But then,

$$[i\hbar\{F,G\}] = \left[\hbar\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right] = [\hbar]\left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right] - [\hbar]\left[\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

We know $[\hbar] = \text{kg.m}^2.\text{s}^{-1} = [q_i p_i]$, and if we make the limits explicit as we did before, it remains from the previous expression:

$$[i\hbar\{F,G\}] = [FG]$$

On the other side:

$$[[F,G]] = [FG - GF]$$

For $FG - GF$ to be well defined, it must be that $[FG] = [GF]$. And so we're done:

$$\boxed{[[F,G]] = [FG] = [i\hbar\{F,G\}]} \quad \square$$