The Theoretical Minimum
Quantum Mechanics - Solutions
L05E01

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Verify this claim.*

The claim being that any 2×2 Hermitian matrix can be represented as a linear combination of:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The general form of a 2×2 Hermitian matrix is:

$$(\forall (r, r', w) \in \mathbb{R}^2 \times \mathbb{C}), \quad \begin{pmatrix} r & w \\ w^* & r' \end{pmatrix}$$

Recall indeed that because for a Hermitian matrix L we have $L = L^\dagger := (L^*)^T$, hence the diagonal elements must be real.

Compare then with the general form for a linear combination of the four matrices above:

$$(\forall (a, b, c, d) \in \mathbb{R}^4), \quad a\sigma_x + b\sigma_y + c\sigma_z + dI = \begin{pmatrix} c+d & a-ib \\ a+ib & c-d \end{pmatrix}$$

Clearly we can identify $w \in \mathbb{C}$ with $a - ib$: this is a general form for a complex number, and this naturally identifies w^* with $a + ib$, as expected.

Regarding the remaining parameters, we have on one side two real parameters, and on the other side, two non-equivalent equations involving two parameters, meaning, two degrees of freedom on both sides. So there's room to identify r with $c+d$ and r' with $c-d$. More precisely, given two arbitrary $(r, r') \in \mathbb{R}^2$, we can always find $(c, d) \in \mathbb{R}^2$ such that $r = c + d$ and $r' = c - d$:

$$\begin{cases} r = c + d \\ r' = c - d \end{cases} \Leftrightarrow \begin{cases} c = r - d \\ d = c - r' \end{cases} \Leftrightarrow \begin{cases} c = r - (c - r') \\ d = (r - d) - r' \end{cases} \Leftrightarrow \begin{cases} c = \frac{r+r'}{2} \\ d = \frac{r-r'}{2} \end{cases}$$

Remark 1. *Note that (real) linear combinations of those 4 matrices are isomorphic to* $\mathbb{Q}$[1].

[1] `https://en.wikipedia.org/wiki/Quaternion`