The Theoretical Minimum

Quantum Mechanics - Solutions

L06E01

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Prove that if $P(a,b)$ factorizes, then the correlation between a and b is zero.*

Let's assume that $P(a,b)$ factorizes, meaning, let's assume than there are two functions P_A and P_B such that:

$$P(a,b) = P_A(a)P_B(b)$$

Recall that the authors have defined the (statistical) correlation the quantity[1]:

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle$$

Where $\langle \sigma_C \rangle$ is the average value of C's observations, also known as the expected value[2], and was defined earlier as:

$$\langle \sigma_C \rangle = \sum_c cP(c)$$

How should we understand $\langle \sigma_A \sigma_B \rangle$? We're trying to find a way to express it as we just did for $\langle \sigma_C \rangle$. It's defined as the average of the product of σ_A and σ_B, meaning, the sum of all possible products of a and b, weighted by some probability distribution, but which one? Well, we don't really know its form specifically, but if for $\langle \sigma_C \rangle$ it was a function of c, then we can guess it must now be a function of a and b: this is the $P(a,b)$ from the exercise statement:

$$\langle \sigma_A \sigma_B \rangle = \sum_a \left(\sum_b abP(a,b) \right)$$

From there, it's just a matter of developing the computation, using our assumption that $P(a,b)$ factorizes:

$$\begin{aligned}
\langle \sigma_A \sigma_B \rangle = &\quad \sum_a \left(\sum_b abP(a,b) \right) \\
= &\quad \sum_a \sum_b (abP_A(a)P_B(b)) \\
= &\quad \sum_a \sum_b ((aP_A(a))(bP_B(b))) \\
= &\quad \left(\sum_a aP_A(a) \right) \left(\sum_b bP_B(b) \right) \\
= &\quad \langle \sigma_A \rangle \langle \sigma_B \rangle
\end{aligned}$$

$$\Leftrightarrow \boxed{\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle = 0} \quad \square$$

Remark 1. *If you're uncertain about the $\sum$ manipulations, you may want to rewrite them as explicit sum over a small number of terms to convince you of their correctness.*

[1] Precise mathematical formulations are more involved, see for instance `https://en.wikipedia.org/wiki/Correlation`

[2] `https://en.wikipedia.org/wiki/Expected_value`