The Theoretical Minimum

Quantum Mechanics - Solutions

L06E02

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Exercise 1. *Show that if the two normalization conditions of Eqs.* 6.4 *are satisfied, then the state-vector of Eq.* 6.5 *is automatically normalized as well. In other words, show that for this product state, normalizing the overall state-vector does not put any additional constraints on the α's and the β's.*

Recall that we're in the context of two distinct state-spaces, each of them referring to a full-blown spin. Spin states for the first space (Alice's) are denoted:

$$\alpha_u|u\} + \alpha_d|d\}, \quad (\alpha_u, \alpha_d) \in \mathbb{C}^2$$

While spin states for the second space (Bob's) are denoted:

$$\beta_u|u\rangle + \beta_d|d\rangle, \quad (\beta_u, \beta_d) \in \mathbb{C}^2$$

Such states are, as usual, normalized: this is the condition referred to by Eqs. 6.4:

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1; \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

The two underlying state spaces (complex space, but really, Hilbert spaces) are glued by a tensor product: this allows the creation of new state space, called the *product state space*, which states can refer to both Alice's and Bob's state in a single expression.

Remark 1. *I encourage you to have a look at how Mathematicians formalize the notion of a tensor product of vector spaces: there is for instance a great introductory YouTube video*[1] *by Michael Penn on the topic.*

The core idea is to start with what is called a formal product of vector spaces, which is a new space built from the span of purely "syntactical" combinations of elements of two (or more) vector spaces. Equivalence classes are then used to constrain this span to be a vector space.

For instance, the three following elements would be distinct elements in the formal product of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$2\begin{pmatrix}1\\2\end{pmatrix} * \begin{pmatrix}3\\4\\5\end{pmatrix}; \qquad \begin{pmatrix}2\\4\end{pmatrix} * \begin{pmatrix}3\\4\\5\end{pmatrix}; \qquad \begin{pmatrix}1\\2\end{pmatrix} * \begin{pmatrix}6\\8\\10\end{pmatrix}$$

But they would be identified by equivalence classes so as to be the same element in the tensor product of $\mathbb{R}^2$ and $\mathbb{R}^3$. We can keep identifying elements likewise until the operations (sum, scalar product) on the formal product space respect the properties the corresponding operations in a vector space.

Here's Eq. 6.5, the general form for such a product state, living in the tensor product space created from Alice's and Bob's state spaces (I've just named it Ψ so as to refer to it later on):

$$|\Psi >= \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

[1] `https://www.youtube.com/watch?v=K7f2pCQ3p3U`

The claim we have to prove is that this vector is naturally normalized, from the normalization constraints imposed on the individual state spaces.
Let's start by computing the norm of product state (assuming an ordered basis $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$:

$$|\Psi|^2 = \langle\Psi|\Psi\rangle = \begin{pmatrix}(\alpha_u\beta_u)^* & (\alpha_u\beta_d)^* & (\alpha_d\beta_u)^* & (\alpha_d\beta_d)^*\end{pmatrix}\begin{pmatrix}\alpha_u\beta_u\\ \alpha_u\beta_d\\ \alpha_d\beta_u\\ \alpha_d\beta_d\end{pmatrix}$$

We can develop it further, using the fact that for $(a,b) \in \mathbb{C}$, $(ab)^* = a^*b^*$:

$$\begin{aligned}
|\Psi|^2 &= \alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d + \alpha_d^*\beta_u^*\alpha_d\beta_u + \alpha_d^*\beta_d^*\alpha_d\beta_d \\
&= \alpha_u^*\alpha_u(\underbrace{\beta_u^*\beta_u + \beta_d^*\beta_d}_{=1}) + \alpha_d^*\alpha_d(\underbrace{\beta_u^*\beta_u + \beta_d^*\beta_d}_{=1}) \\
&= \underbrace{\alpha_u^*\alpha_u + \alpha_d^*\alpha_d}_{=1} \\
&= 1
\end{aligned}$$

But the norm is axiomatically positively defined (i.e. $(\forall\Psi \in \mathcal{H})$, $|\Psi| \geq 0$ with equality iff $\Psi = 0_{\mathcal{H}}$) so:

$$\boxed{|\Psi| = 1} \quad \square$$