The Theoretical Minimum

Quantum Mechanics - Solutions

L06E04

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Use the matrix forms of σ_z, σ_x, and σ_y and the column vectors for $|u\}$ and $|d\}$ to verify Eqs.* 6.6*. Then, use Eqs.* 6.6 *and* 6.7 *to write the equations that were left out of Eqs.* 6.8*. Use the appendix to check your answers.*

As usual, let's recall our Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The base vectors $|u\}$ and $|d\}$ are the canonical basis vectors for $\mathbb{R}^2$:

$$|u\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad |d\} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We're trying to understand how for instance an operator σ_x define on Alice's state spaces can be extended to work on a state vector, taken from a combined state space involving Alice's.

The core idea is that the operator will only act on the "component" of the vector that is related to Alice's state space, while leaving the components involving other state spaces untouched.

Eqs. 6.6 (first column below) simply encode how the spin operators act on the basis vectors, in Alice's state space; Eqs. 6.7 (second column below) are identical, but for Bob's state space:

$$\begin{aligned}
\sigma_z|u\} &= |u\}; & \tau_z|u\rangle &= |u\rangle \\
\sigma_z|d\} &= -|d\}; & \tau_z|d\rangle &= -|d\rangle \\
\sigma_x|u\} &= |d\}; & \tau_x|u\rangle &= |d\rangle \\
\sigma_x|d\} &= |u\}; & \tau_x|d\rangle &= |u\rangle \\
\sigma_y|u\} &= i|d\}; & \tau_y|u\rangle &= i|d\rangle \\
\sigma_y|d\} &= -i|u\}; & \tau_y|d\rangle &= -i|u\rangle
\end{aligned}$$

Now verifying that the matrix products indeed evaluates as such is child's play (matrix $\times$ vector products), there's no use of being more explicit here.

For similar reasons, I'll just write a completed 6.8 here, but won't develop the computations: one just have to follow the aforementioned rule: act with the operator on the correct component, extract the eventual scalar factor, and generally update the corresponding vector component. This yields, in agreement with the appendix:

$$
\begin{array}{rrrrr}
\sigma_z|uu\rangle = & |uu\rangle; & \tau_z|uu\rangle & = & |uu\rangle \\
\sigma_z|ud\rangle = & |ud\rangle; & \tau_z|ud\rangle & = & -|ud\rangle \\
\sigma_z|du\rangle = & -|du\rangle; & \tau_z|du\rangle & = & |du\rangle \\
\sigma_z|dd\rangle = & -|dd\rangle; & \tau_z|dd\rangle & = & -|dd\rangle \\
\hline
\sigma_x|uu\rangle = & |du\rangle; & \tau_x|uu\rangle & = & |ud\rangle \\
\sigma_x|ud\rangle = & |dd\rangle; & \tau_x|ud\rangle & = & |uu\rangle \\
\sigma_x|du\rangle = & |uu\rangle; & \tau_x|du\rangle & = & |dd\rangle \\
\sigma_x|dd\rangle = & |ud\rangle; & \tau_x|dd\rangle & = & |du\rangle \\
\hline
\sigma_y|uu\rangle = & i|du\rangle; & \tau_y|uu\rangle & = & i|ud\rangle \\
\sigma_y|ud\rangle = & i|dd\rangle; & \tau_y|ud\rangle & = & -i|uu\rangle \\
\sigma_y|du\rangle = & -i|uu\rangle; & \tau_y|du\rangle & = & i|dd\rangle \\
\sigma_y|dd\rangle = & -i|ud\rangle; & \tau_y|dd\rangle & = & -i|du\rangle
\end{array}
$$