The Theoretical Minimum

Quantum Mechanics - Solutions

L06E08

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Do the same for the other two entangled triplet states,*

$$|T_2\rangle = \frac{1}{\sqrt{2}}\left(|uu\rangle + |dd\rangle\right)$$

$$|T_3\rangle = \frac{1}{\sqrt{2}}\left(|uu\rangle - |dd\rangle\right)$$

As for previous exercise, this is just about crunching numbers. We won't be using the Pauli matrices explicitly here; instead, we'll use the multiplication table from L06E04

$$\begin{array}{rrrcr}
\sigma_z|uu\rangle = & |uu\rangle; & \tau_z|uu\rangle & = & |uu\rangle \\
\sigma_z|ud\rangle = & |ud\rangle; & \tau_z|ud\rangle & = & -|ud\rangle \\
\sigma_z|du\rangle = & -|du\rangle; & \tau_z|du\rangle & = & |du\rangle \\
\sigma_z|dd\rangle = & -|dd\rangle; & \tau_z|dd\rangle & = & -|dd\rangle \\
\hline
\sigma_x|uu\rangle = & |du\rangle; & \tau_x|uu\rangle & = & |ud\rangle \\
\sigma_x|ud\rangle = & |dd\rangle; & \tau_x|ud\rangle & = & |uu\rangle \\
\sigma_x|du\rangle = & |uu\rangle; & \tau_x|du\rangle & = & |dd\rangle \\
\sigma_x|dd\rangle = & |ud\rangle; & \tau_x|dd\rangle & = & |du\rangle \\
\hline
\sigma_y|uu\rangle = & i|du\rangle; & \tau_y|uu\rangle & = & i|ud\rangle \\
\sigma_y|ud\rangle = & i|dd\rangle; & \tau_y|ud\rangle & = & -i|uu\rangle \\
\sigma_y|du\rangle = & -i|uu\rangle; & \tau_y|du\rangle & = & i|dd\rangle \\
\sigma_y|dd\rangle = & -i|ud\rangle; & \tau_y|dd\rangle & = & -i|du\rangle
\end{array}$$

As the computations are fairly similar, and to save space, I'll be computing the expectation values for T_2 and T_3 in parallel, distinguishing them by a subscript number.

Let's start with $\langle\sigma_z\tau_z\rangle$:

$$\begin{aligned}
\langle\sigma_z\tau_z\rangle_2 &:= \langle T_2|\sigma_z\tau_z|T_2\rangle \\
&= \frac{1}{\sqrt{2}}\langle T_2|\sigma_z\tau_z\left(|uu\rangle + |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_2|\sigma_z\left(|uu\rangle - |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_2|\left(|uu\rangle + |dd\rangle\right) \\
&= \frac{1}{2}(\langle uu| + \langle dd|)(|uu\rangle + |dd\rangle) \\
&= \frac{1}{2}\left(\underbrace{\langle uu|uu\rangle}_{1} + \underbrace{\langle uu|dd\rangle}_{0} + \underbrace{\langle dd|uu\rangle}_{0} + \underbrace{\langle dd|dd\rangle}_{1}\right) \\
&= \boxed{+1}
\end{aligned}$$

$$\begin{aligned}
\langle\sigma_z\tau_z\rangle_3 &:= \langle T_3|\sigma_z\tau_z|T_3\rangle \\
&= \frac{1}{\sqrt{2}}\langle T_3|\sigma_z\tau_z\left(|uu\rangle - |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_3|\sigma_z\left(|uu\rangle + |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_3|\left(|uu\rangle - |dd\rangle\right) \\
&= \frac{1}{2}(\langle uu| - \langle dd|)(|uu\rangle - |dd\rangle) \\
&= \frac{1}{2}\left(\underbrace{\langle uu|uu\rangle}_{1} - \underbrace{\langle uu|dd\rangle}_{0} - \underbrace{\langle dd|uu\rangle}_{0} + \underbrace{\langle dd|dd\rangle}_{1}\right) \\
&= \boxed{+1}
\end{aligned}$$

Moving on to $\langle\sigma_x\tau_x\rangle$:

$$\begin{aligned}
\langle\sigma_x\tau_x\rangle_2 &:= \langle T_2|\sigma_x\tau_x|T_2\rangle \\
&= \frac{1}{\sqrt{2}}\langle T_2|\sigma_x\tau_x\left(|uu\rangle + |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_2|\sigma_x\left(|ud\rangle + |du\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_2|\left(|dd\rangle + |uu\rangle\right) \\
&= \frac{1}{2}(\langle uu| + \langle dd|)(|dd\rangle + |uu\rangle) \\
&= \frac{1}{2}\left(\underbrace{\langle uu|dd\rangle}_{0} + \underbrace{\langle uu|uu\rangle}_{1} + \underbrace{\langle dd|dd\rangle}_{1} + \underbrace{\langle dd|uu\rangle}_{0}\right) \\
&= \boxed{+1}
\end{aligned}$$

$$\begin{aligned}
\langle\sigma_x\tau_x\rangle_3 &:= \langle T_3|\sigma_x\tau_x|T_3\rangle \\
&= \frac{1}{\sqrt{2}}\langle T_3|\sigma_x\tau_x\left(|uu\rangle - |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_3|\sigma_x\left(|ud\rangle - |du\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_3|\left(|dd\rangle - |uu\rangle\right) \\
&= \frac{1}{2}(\langle uu| - \langle dd|)(|dd\rangle - |uu\rangle) \\
&= \frac{1}{2}\left(\underbrace{\langle uu|dd\rangle}_{0} - \underbrace{\langle uu|uu\rangle}_{1} - \underbrace{\langle dd|dd\rangle}_{1} + \underbrace{\langle dd|uu\rangle}_{0}\right) \\
&= \boxed{-1}
\end{aligned}$$

Finally for $\langle\sigma_y\tau_y\rangle$:

$$\begin{aligned}
\langle\sigma_y\tau_y\rangle_2 &:= \langle T_2|\sigma_y\tau_y|T_2\rangle \\
&= \frac{1}{\sqrt{2}}\langle T_2|\sigma_y\tau_y\left(|uu\rangle + |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_2|\sigma_y\left(i|ud\rangle - i|du\rangle\right) \\
&= \frac{i}{\sqrt{2}}\langle T_2|\left(i|dd\rangle + i|uu\rangle\right) \\
&= -\frac{1}{2}(\langle uu| + \langle dd|)(|dd\rangle + |uu\rangle) \\
&= \frac{-1}{2}\left(\underbrace{\langle uu|dd\rangle}_{0} + \underbrace{\langle uu|uu\rangle}_{1} + \underbrace{\langle dd|dd\rangle}_{1} + \underbrace{\langle dd|uu\rangle}_{0}\right) \\
&= \boxed{-1}
\end{aligned}$$

$$\begin{aligned}
\langle\sigma_y\tau_y\rangle_3 &:= \langle T_3|\sigma_y\tau_y|T_3\rangle \\
&= \frac{1}{\sqrt{2}}\langle T_3|\sigma_y\tau_y\left(|uu\rangle - |dd\rangle\right) \\
&= \frac{1}{\sqrt{2}}\langle T_3|\sigma_y\left(i|ud\rangle + i|du\rangle\right) \\
&= \frac{i}{\sqrt{2}}\langle T_3|\left(i|dd\rangle - i|uu\rangle\right) \\
&= -\frac{1}{2}(\langle uu| - \langle dd|)(|dd\rangle - |uu\rangle) \\
&= \frac{-1}{2}\left(\underbrace{\langle uu|dd\rangle}_{0} - \underbrace{\langle uu|uu\rangle}_{1} - \underbrace{\langle dd|dd\rangle}_{1} + \underbrace{\langle dd|uu\rangle}_{0}\right) \\
&= \boxed{+1}
\end{aligned}$$

We can conclude, from those expectation values alone, that whenever:

- The expectation value is -1, Bob and Alice measure a spin pointing in different directions;
- The expectation value is $+1$, Bob and Alice measure a spin pointing in the same direction.

I just want to spend a few more lines to make something clear. Recall the definition of $|\text{sing}\rangle$:

$$|\text{sing}\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

The argument of the authors was that, the reason for $\langle\tau_z\sigma_z\rangle$ to be -1 was that $|\text{sing}\rangle$ is built from two spins, one of which is always up while the other is down, and we're measuring both spin alongside the axis on which they are either up or down.

However, in the case of e.g. $\langle\tau_x\sigma_x\rangle$, the answer was not as obviously, because we're in this case measuring the spins alongside the x-axis, and it's not immediate from the expression of $|\text{sing}\rangle$ what kind of balance we have alongside the x-axis.

Let's do a little experiment. Recall the definition of the "basis vectors" for the x-axis, left and right:

$$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle); \qquad |l\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$$

We want to express, say, T_3 in terms of $|l\rangle$ and $|r\rangle$, to see if indeed, when expressed as such, T_3 is created from two spins such that when one is left, the other is right, which would be concordant with the idea that $\langle\sigma_x\tau_x\rangle_3 = -1$. Let's start by rewriting $|u\rangle$ and $|d\rangle$ in terms of $|r\rangle$ and $|l\rangle$:

$$\begin{cases} |r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle) \\ |l\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle) \end{cases} \Leftrightarrow \begin{cases} |u\rangle = \sqrt{2}|r\rangle - |d\rangle \\ |d\rangle = -\sqrt{2}|l\rangle + |u\rangle \end{cases} \Leftrightarrow \begin{cases} |u\rangle = \frac{\sqrt{2}}{2}(|r\rangle + |l\rangle) \\ |d\rangle = \frac{\sqrt{2}}{2}(|r\rangle - |l\rangle) \end{cases}$$

Let's now rewrite T_3 in the $|r\rangle$, $|l\rangle$ basis:

$$\begin{aligned} |T_3\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\ &= \frac{1}{\sqrt{2}}(|u\} \otimes |u\rangle - |d\} \otimes |d\rangle) \\ &= \frac{1}{\sqrt{2}}\left(\frac{2}{4}(|r\} + |l\})(|r\rangle + |l\rangle) - \frac{2}{4}(|r\} - |l\})(|r\rangle - |l\rangle)\right) \\ &= \frac{1}{2\sqrt{2}}(|rr\rangle + |rl\rangle + |lr\rangle + |ll\rangle - (|rr\rangle - |rl\rangle - |lr\rangle + |ll\rangle)) \\ &= \frac{1}{\sqrt{2}}(|rl\rangle + |lr\rangle) \end{aligned}$$

And indeed, as expected, T_3 is built from two spins such that when one is left, the other is right. Let's do another one to be sure: consider T_2 on the x-axis: this gives us a $+1$, so we expect a normalized linear combination of $|rr\rangle$ and $|ll\rangle$.

$$\begin{aligned} |T_2\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\ &= \frac{1}{\sqrt{2}}(|u\} \otimes |u\rangle + |d\} \otimes |d\rangle) \\ &= \frac{1}{\sqrt{2}}\left(\frac{2}{4}(|r\} + |l\})(|r\rangle + |l\rangle) + \frac{2}{4}(|r\} - |l\})(|r\rangle - |l\rangle)\right) \\ &= \frac{1}{2\sqrt{2}}(|rr\rangle + |rl\rangle + |lr\rangle + |ll\rangle + (|rr\rangle - |rl\rangle - |lr\rangle + |ll\rangle)) \\ &= \frac{1}{\sqrt{2}}(|rr\rangle + |ll\rangle) \end{aligned}$$