The Theoretical Minimum

Quantum Mechanics - Solutions

L06E10

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *A system of two spins has the Hamiltonian*

$$\boldsymbol{H} = \frac{\omega}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$$

What are the possible energies of the system, and what are the eigenvectors of the Hamiltonian?

Suppose the system starts in the state $|uu\rangle$. What is the state at any later time? Answer the same question for initial states $|ud\rangle$, $|du\rangle$, and $|dd\rangle$.

The first part of the question essentially is about diagonalizing the Hamiltonian: the eigenvalues correspond to the measurable values for the energy. More generally, the exercise is about repeating what we've done earlier in chapter 4, in particular in exercise L04E06, meaning, applying what the authors call the *recipe for a Schrödinger Ket (section* 4.13*)*:

1. Derive, look up, guess, borrow, or steal the Hamiltonian operator H;
2. Prepare an initial state $|\Psi(0)\rangle$;
3. Find the eigenvalues and eigenvectors of H by solving the time-independent Schrödinger equation,
$$H|E_j\rangle = E_j|E_j\rangle$$
4. Use the initial state-vector $|\Psi(0)\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3, to calculate the initial coefficients $\alpha_j(0)$:
$$\alpha_j(0) = \langle E_j|\Psi(0)\rangle$$
5. Rewrite $|\Psi(0)\rangle$ in terms of the eigenvectors $|E_j\rangle$ and the initial coefficients $\alpha_j(0)$:
$$|\Psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle$$
6. In the above equation, replace each $\alpha_j(0)$ with $\alpha_j(t)$ to capture its time-dependence. As a result, $|\Psi(0)\rangle$ becomes $|\Psi(t)\rangle$:
$$|\Psi(t)\rangle = \sum_j \alpha_j(t)|E_j\rangle$$
7. Using Eq. 4.30[1], replace each $\alpha_j(t)$ with $\alpha_j(0)\exp(-\frac{i}{\hbar}E_j t)$:
$$|\Psi(t)\rangle = \sum_j \alpha_j(0)\exp(-\frac{i}{\hbar}E_j t)|E_j\rangle$$

[1]This equation corresponds exactly to what this step describes

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We'll start by diagonalizing $\boldsymbol{H}$, and then, by loosely applying the rest of the procedure with the various proposed initial states. Recall from the previous exercise that we've found 4 eigenvectors for $\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$:

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|\text{sing}\rangle &= -3|\text{sing}\rangle \\
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|T_1\rangle &= +1|T_1\rangle \\
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|T_2\rangle &= +1|T_2\rangle \\
(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})|T_3\rangle &= +1|T_3\rangle
\end{aligned}$$

Let's recall the expression of those 4 vectors in the up/down basis:

$$\begin{aligned}
|\text{sing}\rangle &= \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle); \qquad & |T_1\rangle &= \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \\
|T_2\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle); \qquad & |T_3\rangle &= \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)
\end{aligned}$$

It is immediate to check that those eigenvectors all have norm 1, and that they are orthogonal pairwise[2].

Furthermore, we know that $\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$ is an operator in a 4 dimensional vector space $A \otimes B$[3]. And we know from the spectral theorem (aka, the fundamental theorem, proved in L03E01) that the eigenvectors of a Hermitian operator (i.e. an observable) make an orthonormal basis for the surrounding vector space.

Hence we can conclude that our 4 eigenvectors $|\text{sing}\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ are *the* eigenvectors of $\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$: there are no others, for we've reached the dimension of our vector space $A \otimes B$. By scaling our operator by $\omega/2$, we find back our Hamiltonian $\boldsymbol{H}$, for which we then have the same eigenvectors, only the eigenvalues now need to be shifted likewise:

$$\begin{aligned}
\boldsymbol{H}|\text{sing}\rangle &= \frac{-3\omega}{2}|\text{sing}\rangle; \qquad & \boldsymbol{H}|T_1\rangle &= \frac{+\omega}{2}|T_1\rangle \\
\boldsymbol{H}|T_2\rangle &= \frac{+\omega}{2}|T_2\rangle; \qquad & \boldsymbol{H}|T_3\rangle &= \frac{+\omega}{2}|T_3\rangle
\end{aligned}$$

Hence, we can only measure two values for the energy:

$$\boxed{E_{\text{sing}} = \frac{-3\omega}{2}; \qquad E_{T_1} = E_{T_2} = E_{T_3} = \frac{+\omega}{2}}$$

And our eigenvectors are:

$$\boxed{|\text{sing}\rangle, \quad |T_1\rangle, \quad |T_2\rangle, \quad |T_3\rangle}$$

At this point, we've reached the end of step 3. of the *recipe for a Schrödinger cat* recalled earlier. We're now ready to follow through the other steps, by varying the initial state. Let's start as suggested with $|\Psi_{uu}(0)\rangle = |uu\rangle$: we're trying to rewrite this initial vector state in the basis corresponding to the eigenvectors of our observable (our Hamiltonian).

To this effect, we start by computing the coefficient $\alpha_j(0)$:

$$\begin{aligned}
\alpha_{\text{sing}}(0) &:= \langle\text{sing}|\Psi_{uu}(0)\rangle \qquad & \alpha_{T_1}(0) &:= \langle T_1|\Psi_{uu}(0)\rangle \\
&= \langle\text{sing}|uu\rangle & &= \langle T_1|uu\rangle \\
&= \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|uu\rangle & &= \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|uu\rangle \\
&= \boxed{0} & &= \boxed{0}
\end{aligned}$$

[2]If unsure, compute respectively the norm, which is derived from the inner-product: $||\Psi\rangle| := \sqrt{\langle\Psi|\Psi\rangle}$, and that the same inner-product between two vectors is zero iff said vectors are orthogonal

[3]If this is unclear, you can refer to the beginning on this Chapter (6), where we explore how the combine vector space was built

$$\begin{aligned} \alpha_{T_2}(0) &:= \langle T_2|\Psi_{uu}(0)\rangle \\ &= \langle T_2|uu\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|)|uu\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}} \end{aligned} \qquad \begin{aligned} \alpha_{T_3}(0) &:= \langle T_3|\Psi_{uu}(0)\rangle \\ &= \langle T_3|uu\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)|uu\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}} \end{aligned}$$

Hence we can rewrite (step 5.) $|\Psi_{uu}(0)\rangle = |uu\rangle$ in the eigenbase:

$$|\Psi_{uu}(0)\rangle = |uu\rangle = \sum_j \alpha_j(0)|E_j\rangle = \frac{1}{\sqrt{2}}(|T_2\rangle + |T_3\rangle)$$

And from a previous equation (4.30) we can find the evolution over time of our state:

$$|\Psi_{uu}(t)\rangle = \sum_j \alpha_j(0)\exp(-\frac{i}{\hbar}E_j t)|E_j\rangle$$

That is:

$$\boxed{|\Psi_{uu}(t)\rangle = \frac{1}{\sqrt{2}}\exp(-\frac{\omega i}{2\hbar}t)(|T_2\rangle + |T_3\rangle)}$$

Let's repeat the exact same process, but this time with an initial state $|\Psi_{ud}(0)\rangle = |ud\rangle$. I'll just perform the computation, you can refer to the previous steps if need be.

$$\begin{aligned} \alpha_{\text{sing}}(0) &:= \langle \text{sing}|\Psi_{ud}(0)\rangle \\ &= \langle \text{sing}|ud\rangle \\ &= \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|ud\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}} \end{aligned} \qquad \begin{aligned} \alpha_{T_1}(0) &:= \langle T_1|\Psi_{ud}(0)\rangle \\ &= \langle T_1|ud\rangle \\ &= \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|ud\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}} \end{aligned}$$

$$\begin{aligned} \alpha_{T_2}(0) &:= \langle T_2|\Psi_{ud}(0)\rangle \\ &= \langle T_2|ud\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|)|ud\rangle \\ &= \boxed{0} \end{aligned} \qquad \begin{aligned} \alpha_{T_3}(0) &:= \langle T_3|\Psi_{ud}(0)\rangle \\ &= \langle T_3|ud\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)|ud\rangle \\ &= \boxed{0} \end{aligned}$$

But:

$$|\Psi_{ud}(t)\rangle = \sum_j \alpha_j(0)\exp(-\frac{i}{\hbar}E_j t)|E_j\rangle$$

So:

$$\boxed{|\Psi_{ud}(t)\rangle = \frac{1}{\sqrt{2}}\left(\exp(\frac{3\omega i}{2\hbar}t)|\text{sing}\rangle + \exp(-\frac{\omega i}{2\hbar}t)|T_1\rangle\right)}$$

Let's do it more time, with an initial state of $|\Psi_{du}(0)\rangle = |du\rangle$.

$$\begin{aligned} \alpha_{\text{sing}}(0) &:= \langle \text{sing}|\Psi_{du}(0)\rangle \\ &= \langle \text{sing}|du\rangle \\ &= \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|du\rangle \\ &= \boxed{-\frac{1}{\sqrt{2}}} \end{aligned} \qquad \begin{aligned} \alpha_{T_1}(0) &:= \langle T_1|\Psi_{du}(0)\rangle \\ &= \langle T_1|du\rangle \\ &= \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|du\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}} \end{aligned}$$

$$\begin{aligned} \alpha_{T_2}(0) &:= \langle T_2|\Psi_{du}(0)\rangle \\ &= \langle T_2|du\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|)|du\rangle \\ &= \boxed{0} \end{aligned} \qquad \begin{aligned} \alpha_{T_3}(0) &:= \langle T_3|\Psi_{du}(0)\rangle \\ &= \langle T_3|du\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)|du\rangle \\ &= \boxed{0} \end{aligned}$$

But:

$$|\Psi_{du}(t)\rangle = \sum_j \alpha_j(0)\exp(-\frac{i}{\hbar}E_j t)|E_j\rangle$$

So:

$$\boxed{|\Psi_{du}(t)\rangle = \frac{1}{\sqrt{2}}\left(\exp(-\frac{\omega i}{2\hbar}t)|T_1\rangle - \exp(\frac{3\omega i}{2\hbar}t)|\text{sing}\rangle\right)}$$

One last time, starting from $|\Psi_{dd}(0)\rangle = |dd\rangle$.

$$\begin{aligned} \alpha_{\text{sing}}(0) &:= \langle \text{sing}|\Psi_{dd}(0)\rangle \\ &= \langle \text{sing}|dd\rangle \\ &= \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|dd\rangle \\ &= \boxed{0} \end{aligned} \qquad \begin{aligned} \alpha_{T_1}(0) &:= \langle T_1|\Psi_{dd}(0)\rangle \\ &= \langle T_1|dd\rangle \\ &= \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|dd\rangle \\ &= \boxed{0} \end{aligned}$$

$$\begin{aligned} \alpha_{T_2}(0) &:= \langle T_2|\Psi_{dd}(0)\rangle \\ &= \langle T_2|dd\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|)|dd\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}} \end{aligned} \qquad \begin{aligned} \alpha_{T_3}(0) &:= \langle T_3|\Psi_{dd}(0)\rangle \\ &= \langle T_3|dd\rangle \\ &= \frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)|dd\rangle \\ &= \boxed{-\frac{1}{\sqrt{2}}} \end{aligned}$$

But:

$$|\Psi_{dd}(t)\rangle = \sum_j \alpha_j(0)\exp(-\frac{i}{\hbar}E_j t)|E_j\rangle$$

So:

$$\boxed{|\Psi_{dd}(t)\rangle = \frac{1}{\sqrt{2}}\exp(-\frac{\omega i}{\hbar}t)\,(|T_2\rangle - |T_3\rangle)}$$