

The Theoretical Minimum

Quantum Mechanics - Solutions

L07E02

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. Calculate the matrix elements of $\sigma_z \otimes \tau_x$ by forming inner products as we did in Eq. 7.2.

This is essentially the same exercise as the previous one, but with a different composite operator. To check for errors, I'll still do the computation using the two approaches.

We'll start with the approach suggested in the exercise's statement: let's first start by recalling the portion of interest from the multiplication table computed in L06E04:

$$\begin{aligned}\sigma_z|uu\rangle &= |uu\rangle; & \tau_x|uu\rangle &= |ud\rangle \\ \sigma_z|ud\rangle &= |ud\rangle; & \tau_x|ud\rangle &= |uu\rangle \\ \sigma_z|du\rangle &= -|du\rangle; & \tau_x|du\rangle &= |dd\rangle \\ \sigma_z|dd\rangle &= -|dd\rangle; & \tau_x|dd\rangle &= |du\rangle\end{aligned}$$

Then, Eq. 7.2 applied to $\sigma_z \otimes \tau_x$ will give:

$$\begin{aligned}\sigma_z \otimes \tau_x &= \begin{pmatrix} \langle uu|(\sigma_z \otimes \tau_x)|uu\rangle & \langle uu|(\sigma_z \otimes \tau_x)|ud\rangle & \langle uu|(\sigma_z \otimes \tau_x)|du\rangle & \langle uu|(\sigma_z \otimes \tau_x)|dd\rangle \\ \langle ud|(\sigma_z \otimes \tau_x)|uu\rangle & \langle ud|(\sigma_z \otimes \tau_x)|ud\rangle & \langle ud|(\sigma_z \otimes \tau_x)|du\rangle & \langle ud|(\sigma_z \otimes \tau_x)|dd\rangle \\ \langle du|(\sigma_z \otimes \tau_x)|uu\rangle & \langle du|(\sigma_z \otimes \tau_x)|ud\rangle & \langle du|(\sigma_z \otimes \tau_x)|du\rangle & \langle du|(\sigma_z \otimes \tau_x)|dd\rangle \\ \langle dd|(I \otimes \tau_x)|uu\rangle & \langle dd|(\sigma_z \otimes \tau_x)|ud\rangle & \langle dd|(\sigma_z \otimes \tau_x)|du\rangle & \langle dd|(\sigma_z \otimes \tau_x)|dd\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle uu|\sigma_z|ud\rangle & \langle uu|\sigma_z|uu\rangle & \langle uu|\sigma_z|dd\rangle & \langle uu|\sigma_z|du\rangle \\ \langle ud|\sigma_z|ud\rangle & \langle ud|\sigma_z|uu\rangle & \langle ud|\sigma_z|dd\rangle & \langle ud|\sigma_z|du\rangle \\ \langle du|\sigma_z|ud\rangle & \langle du|\sigma_z|uu\rangle & \langle du|\sigma_z|dd\rangle & \langle du|\sigma_z|du\rangle \\ \langle dd|\sigma_z|ud\rangle & \langle dd|\sigma_z|uu\rangle & \langle dd|\sigma_z|dd\rangle & \langle dd|\sigma_z|du\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle uu|ud\rangle & \langle uu|uu\rangle & -\langle uu|dd\rangle & -\langle uu|du\rangle \\ \langle ud|ud\rangle & \langle ud|uu\rangle & -\langle ud|dd\rangle & -\langle ud|du\rangle \\ \langle du|ud\rangle & \langle du|uu\rangle & -\langle du|dd\rangle & -\langle du|du\rangle \\ \langle dd|ud\rangle & \langle dd|uu\rangle & -\langle dd|dd\rangle & -\langle dd|du\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\end{aligned}$$

Let's verify our computation using the second approach, relying on Eq. 7.6 of the book:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}$$

Recall the Pauli matrices:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Which then yields:

$$\begin{aligned}\sigma_z \otimes \tau_x &= \begin{pmatrix} 1 \times \tau_x & 0 \times \tau_x \\ 0 \times \tau_x & -1 \times \tau_x \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}\end{aligned}$$

Which agrees with our previous result.