# The Theoretical Minimum

## Quantum Mechanics - Solutions

### L07E04

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

M. Bivert

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**Exercise 1.** *Calculate the density matrix for:*

$$|\Psi\rangle = \alpha|u\rangle + \beta|d\rangle$$

*Answer:*

$$\begin{aligned} \psi(u) = \quad & \alpha; \qquad & \psi^*(u) = \quad & \alpha^* \\ \psi(d) = \quad & \beta; \qquad & \psi^*(d) = \quad & \beta^* \end{aligned}$$

$$\rho_{a'a} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\beta \\ \beta^*\alpha & \beta^*\beta \end{pmatrix}$$

*Now try plugging in some numbers for $\alpha$ and $\beta$. Make sure they are normalized to 1. For example, $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{2}}$.*

Start by recalling the definition of the density matrix for a single spin in a known state:

$$\rho_{aa'} = \psi^*(a')\psi(a)$$

Now we have no wave function $\psi$ in the exercise statement (the answer set aside), but we can find it by identification with general form of $|\Psi\rangle$:

$$|\Psi\rangle = \sum_{a,b,c,\dots} \psi(a, b, c, \dots)|a, b, c, \dots\rangle$$

Hence, $\psi(u)$ is the component of $|\Psi\rangle$ following the $|u\rangle$ axis, and $\psi(d)$ the one on the $|d\rangle$ axis:

$$\psi(u) = \langle u|\Psi\rangle = \alpha; \qquad \psi(d) = \langle d|\Psi\rangle = \beta;$$

Immediately:

$$\psi^*(u) = \alpha^*; \qquad \psi^*(d) = \beta^*;$$

Then it's just about packaging all the $\rho_{aa'}$ in a matrix: the basis is ordered $(\{|u\rangle, |d\rangle\})$ hence:

$$\rho = \begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix} = \begin{pmatrix} \psi^*(u)\psi(u) & \psi^*(d)\psi(u) \\ \psi^*(u)\psi(d) & \psi^*(d)\psi(d) \end{pmatrix} = \boxed{\begin{pmatrix} \alpha^*\alpha & \beta^*\alpha \\ \alpha^*\beta & \beta^*\beta \end{pmatrix}}$$

**Remark 1.** *We could also use the fact that the density operator is defined as a linear combination of of projectors corresponding to the potential states of the system, each scaled by a probability, and so that the sum of those probabilities is 1, e.g.:*

$$\rho = \sum_i P_i|\psi_i\rangle\langle\psi_i|; \qquad \textit{where: } \sum_i P_i = 1$$

*As we're in the case of a single spin in a known state $|\Psi\rangle$, this reduces to*

$$\rho = 1|\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi|$$

*Assuming again the ordered basis $\{|u\rangle, |d\rangle\}$, we can write $\langle\Psi|$ and $|\Psi\rangle$ in column form, and perform the outer-product:*

$$\rho = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

This allows us to double-check our previous result: it seems there's a typo in the exercise statement.

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Let's compute a few density matrices for well-known states:

$$|u\rangle = 1|u\rangle + 0|d\rangle \quad \Rightarrow \quad \rho_{|u\rangle} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|d\rangle = 0|u\rangle + 1|d\rangle \quad \Rightarrow \quad \rho_{|d\rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|r\rangle} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|l\rangle} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|i\rangle} = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}$$

$$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \quad \Rightarrow \quad \rho_{|o\rangle} = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}$$

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The French version of this exercise[1] is a bit more interesting, there are a few additional questions. We can for instance check that $\rho$ is Hermitian:

$$\rho^\dagger = (\rho^*)^T = \begin{pmatrix} (\alpha^*\alpha)^* & (\beta^*\alpha)^* \\ (\alpha^*\beta)^* & (\beta^*\beta)^* \end{pmatrix}^T = \begin{pmatrix} \alpha\alpha^* & \beta\alpha^* \\ \alpha\beta^* & \beta\beta^* \end{pmatrix}^T = \begin{pmatrix} \alpha^*\alpha & \beta^*\alpha \\ \alpha^*\beta & \beta^*\beta \end{pmatrix} =: \rho \quad \square$$

Or that its trace is 1, because of the normalization condition on $|\Psi\rangle$:

$$\mathrm{Tr}(\rho) = \alpha^*\alpha + \beta^*\beta = 1 \quad \square$$

Finally, we can check that $\rho$ projects to $|\Psi\rangle$. Consider a vector which has a component perpendicular to $|\Psi\rangle$, that is, in the direction of $|\Psi^\perp\rangle$, and a component in the direction of $|\Psi\rangle$

$$|\Phi\rangle = \gamma|\Psi^\perp\rangle + \delta|\Psi\rangle$$

By linearity:

$$\rho|\Phi\rangle = \gamma\rho|\Psi^\perp\rangle + \delta\rho|\Psi\rangle$$

Using the fact that $\rho = |\Psi\rangle\langle\Psi|$, we see, by associativity on the products, and by the orthogonality condition between $|\Psi\rangle$ and $|\Psi^\perp\rangle$:

$$\rho|\Psi^\perp\rangle = (|\Psi\rangle\langle\Psi|)|\Psi^\perp\rangle = |\Psi\rangle(\underbrace{\langle\Psi|\Psi^\perp\rangle}_{=0}) = 0$$

On the other hand, by the normalization condition on $|\Psi\rangle$:

$$\rho|\Psi\rangle = (|\Psi\rangle\langle\Psi|)|\Psi^\rangle = |\Psi\rangle(\underbrace{\langle\Psi|\Psi\rangle}_{=1}) = |\Psi\rangle$$

By injecting the two previous results in the one before, it follows that indeed that $\rho$ projects a vector on the $|\Psi\rangle$ direction:

$$\rho|\Phi\rangle = \delta|\Psi\rangle$$

[1] See `https://leminimumtheorique.jimdofree.com/le%C3%A7on-7/exercice-7-4/` for a relevant excerpt, which by the way seems to confirm the typo hypothesis.