The Theoretical Minimum

Quantum Mechanics - Solutions

L07E08

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Exercise 1. *Consider the following states*

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{2}\left(|uu\rangle + |ud\rangle + |du\rangle + |dd\rangle\right) \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}\left(|uu\rangle + |dd\rangle\right) \\ |\psi_3\rangle &= \frac{1}{5}\left(3|uu\rangle + 4|ud\rangle\right) \end{aligned}$$

For each one, calculate Alice's density matrix, and Bob's density matrix. Check their properties.

Let's recall first the definition of the matrix elements for Alice's density matrix, and second, by symmetry, Bob's:

$$\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b); \qquad \rho_{b'b} = \sum_a \psi^*(a,b)\psi(a,b')$$

Let's start with $|\psi_1\rangle$. We know Alice's matrix must be of the form:

$$\rho_A = \begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix}$$

And so must be Bob's actually. Filling in with our previous formulas, we obtain:

$$\begin{aligned} \rho_{1A} &= \begin{pmatrix} \psi_1^*(u,u)\psi_1(u,u) + \psi_1^*(u,d)\psi_1(u,d) & \psi_1^*(d,u)\psi_1(u,u) + \psi_1^*(d,d)\psi_1(u,d) \\ \psi_1^*(u,u)\psi_1(d,u) + \psi_1^*(u,d)\psi_1(d,d) & \psi_1^*(d,u)\psi_1(d,u) + \psi_1^*(d,d)\psi_1(d,d) \end{pmatrix} \\ &= \begin{pmatrix} (1/2)(1/2) + (1/2)(1/2) & (1/2)(1/2) + (1/2)(1/2) \\ (1/2)(1/2) + (1/2)(1/2) & (1/2)(1/2) + (1/2)(1/2) \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}} \end{aligned}$$

Where, remember, the wave function's values correspond to the basis vector coefficients, which are all 1/2 here. By symmetry, we would obtain exactly the same matrix for Bob:

$$\rho_{1B} = \boxed{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}}$$

Let's check the density matrices properties:

- Clearly, $\rho_{1A} = \rho_{1B}$ is Hermitian;
- Its trace is $1/2 + 1/2 = 1$, as expected;

- Let's compute its square:

$$\rho_{1A}^2 = \rho_{1B}^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \rho_{1A} = \rho_{1B}$$

  And $\mathrm{Tr}(\rho_{1A}^2) = \mathrm{Tr}(\rho_{1B}^2) = 1$, from which we can conclude that $\boxed{\psi_1 \text{ is a pure state.}}$

- Without having to compute them explicitly, this implies that its eigenvalues must be 0 and 1.

Let's compute the eigenvalues by partially diagonalizing the matrix anyway for practice: an eigenvector $|\lambda\rangle$ is tied to an eigenvalue λ by:

$$\rho_{1A}|\lambda\rangle = \lambda|\lambda\rangle \Leftrightarrow \rho_{1A}|\lambda\rangle - \lambda|\lambda\rangle = 0 \Leftrightarrow (\rho_{1A} - \lambda I)|\lambda\rangle = 0$$

Because an eigenvector is by definition non-zero, this implies that $\rho_{1A} - \lambda I$ must be non-invertible[1]. This implies that:

$$\det(\rho_{1A} - \lambda I) = 0 \Leftrightarrow 0 = \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda)^2 - \frac{1}{2}^2\right) = (\frac{1}{2} - \lambda - \frac{1}{2})(\frac{1}{2} - \lambda + \frac{1}{2}) = \lambda(\lambda - 1)$$

$$\Leftrightarrow \boxed{\begin{cases} \lambda = 0 \\ \lambda = 1 \end{cases}}$$

As expected.

Let's move on to ψ_2: by a similar reasoning as before we have:

$$\begin{aligned} \rho_{2A} = & \begin{pmatrix} \psi_2^*(u,u)\psi_2(u,u) + \psi_2^*(u,d)\psi_2(u,d) & \psi_2^*(d,u)\psi_2(u,u) + \psi_2^*(d,d)\psi_2(u,d) \\ \psi_2^*(u,u)\psi_2(d,u) + \psi_2^*(u,d)\psi_2(d,d) & \psi_2^*(d,u)\psi_2(d,u) + \psi_2^*(d,d)\psi_2(d,d) \end{pmatrix} \\ = & \begin{pmatrix} (1/\sqrt{2})(1/\sqrt{2}) + (0)(0) & (0)(1/\sqrt{2}) + (1/\sqrt{2})(0) \\ (1/\sqrt{2})(0) + (0)(1/\sqrt{2}) & (0)(0) + (1/\sqrt{2})(1/\sqrt{2}) \end{pmatrix} \\ = & \boxed{\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}} \end{aligned}$$

Again, by a symmetry argument, we can already conclude that $\rho_{2B} = \rho_{2A}$ (the idea is that you can swap the labels corresponding to Bob and Alice in the description of the state ψ_2 and by reordering the terms, you see that the state is unchanged).

Finally, let's check the density matrices properties:

1. Clearly Hermitian;
2. $\mathrm{Tr}(\rho_{2A}) = 1/2 + 1/2 = 1$;
3. Let's compute the square to determine the state quality:

$$\rho_{2A}^2 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \neq \rho_{2A}$$

   and $\mathrm{Tr}(\rho_{2A}^2) = 1/2 < 1$: $\boxed{\psi_2 \text{ is a mixed state}}$;
4. The matrix is diagonal: clearly, all its eigenvalue (there's a single degenerate eigenvalue 1/2) are positive and ≤ 1.

Moving on to the last one. Observe that this time, there is not symmetry between Alice and Bob matrices, so we'll have to compute them both.

[1] For otherwise, multiply both sides of the equation by its inverse: LHS is equal to $|\lambda\rangle$ while the RHS is still equal to 0

$$
\begin{aligned}
\rho_{3A} = & \begin{pmatrix} \psi_3^*(u,u)\psi_3(u,u) + \psi_3^*(u,d)\psi_3(u,d) & \psi_3^*(d,u)\psi_3(u,u) + \psi_3^*(d,d)\psi_3(u,d) \\ \psi_3^*(u,u)\psi_3(d,u) + \psi_3^*(u,d)\psi_3(d,d) & \psi_3^*(d,u)\psi_3(d,u) + \psi_3^*(d,d)\psi_3(d,d) \end{pmatrix} \\
= & \begin{pmatrix} (3/5)(3/5) + (4/5)(4/5) & (0)(3/5) + (0)(4/5) \\ (3/5)(0) + (4/5)(0) & (0)(0) + (0)(0) \end{pmatrix} \\
= & \begin{pmatrix} 9/25 + 16/25 & 0 \\ 0 & 0 \end{pmatrix} \\
= & \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}
\end{aligned}
$$

Regarding density matrices properties:

1. Hermitian;
2. $\text{Tr}(\rho_{3A}) = 1 + 0 = 1$;
3. $\rho_{3A}^2 = \rho_{3A}$: $\boxed{\psi_3 \text{ is a pure state}}$;
4. This is confirmed by the eigenvalues 1 and 0 (matrix trivially diagonal).

Remains Bob's matrix!

$$
\begin{aligned}
\rho_{3B} = & \begin{pmatrix} \psi_3^*(u,u)\psi_3(u,u) + \psi_3^*(d,u)\psi_3(d,u) & \psi_3^*(u,u)\psi_3(u,d) + \psi_3^*(d,u)\psi_3(d,d) \\ \psi_3^*(u,d)\psi_3(u,u) + \psi_3^*(d,d)\psi_3(d,u) & \psi_3^*(u,d)\psi_3(u,d) + \psi_3^*(d,d)\psi_3(d,d) \end{pmatrix} \\
= & \begin{pmatrix} (3/5)(3/5) + (0)(0) & (3/5)(4/5) + (0)(0) \\ (4/5)(3/5) + (0)(0) & (4/5)(4/5) + (0)(0) \end{pmatrix} \\
= & \boxed{\frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}}
\end{aligned}
$$

One last time, let's check its density matrices properties:

1. Clearly Hermitian;
2. $\text{Tr}(\rho_{3B}) = 9/25 + 16/25 = 1$;
3. Let's square it to determine the state quality:

$$
\begin{aligned}
\rho_{3B}^2 = & \frac{1}{25^2}\begin{pmatrix} 9 \times 9 + 12 \times 12 & 9 \times 12 + 12 \times 16 \\ 12 \times 9 + 16 \times 12 & 12 \times 12 + 16 \times 16 \end{pmatrix} \\
= & \frac{1}{25^2}\begin{pmatrix} 81 + 100 + 40 + 4 & 90 + 18 + 100 + 80 + 12 \\ 90 + 18 + 100 + 80 + 12 & 100 + 40 + 4 + 100 + 120 + 36 \end{pmatrix} \\
= & \frac{1}{25^2}\begin{pmatrix} 225 & 300 \\ 300 & 400 \end{pmatrix} \\
= & \frac{1}{25^2}\begin{pmatrix} (4 \times 2 + 1) \times 25 & 3 \times 4 \times 25 \\ 3 \times 4 \times 25 & 4 \times 4 \times 25 \end{pmatrix} \\
= & \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} = \rho_{3B}
\end{aligned}
$$

   Thus $\text{Tr}(\rho_{3B}^2) = \text{Tr}(\rho_{3B}) = 1$ and $\boxed{\psi_3 \text{ is a pure state}}$;

4. This implies again that its eigenvalues must be 0 and 1

Let's compute the eigenvalues for practice, going a bit faster this time:

$$
\begin{vmatrix} 9/25 - \lambda & 12/25 \\ 12/25 & 16/25 - \lambda \end{vmatrix} = 0 \Leftrightarrow \left(\left(\frac{9}{25} - \lambda\right)\left(\frac{16}{25} - \lambda\right) - \left(\frac{12}{25}\right)^2 \right) = 0
$$

$$
\Leftrightarrow \lambda^2 - \lambda + \frac{9 \times 16}{25^2} - \left(\frac{12}{25}\right)^2 = 0
$$

$$\Leftrightarrow \lambda^2 - \lambda + \frac{3 \times 3 \times 4 \times 4}{25^2} - \frac{3 \times 4 \times 3 \times 4}{25^2} = 0$$

$$\Leftrightarrow \lambda(\lambda - 1) = 0$$

$$\Leftrightarrow \boxed{\begin{cases} \lambda = 0 \\ \lambda = 1 \end{cases}}$$