The Theoretical Minimum

Quantum Mechanics - Solutions

L07E09

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Exercise 1. *Given any Alice observable* $\boldsymbol{A}$ *and Bob observable* $\boldsymbol{B}$*, show that for a product state, the correlation* $C(A, B)$ *is zero.*

Recall that we're in the context of a composite system S_{AB} made from two state spaces, one corresponding to Alice, S_A, and one corresponding to Bob, S_B, mathematically tied by a tensor product.

The correlation $C(\boldsymbol{A}, \boldsymbol{B})$ between two observables $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as[1]:

$$C(\boldsymbol{A}, \boldsymbol{B}) := \langle \boldsymbol{AB} \rangle - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle$$

Remember that the authors proved[2] that the expected value $\langle \boldsymbol{L} \rangle$ of an observable $\boldsymbol{L}$ being in a state $|\Psi\rangle$ is:

$$\langle \boldsymbol{L} \rangle = \langle \Psi | \boldsymbol{L} | \Psi \rangle$$

Remark 1. *There's an issue in this first derivation, reported by Jannis Koeckeritz; I've left it so you can "have fun" trying to find it on your own; the solution is in this footnote*[3]*.*

Here's a first derivation, where we use the following formula[4] defined for an observable $\boldsymbol{L}$, and a system described by a density matrix ρ:

$$\langle \boldsymbol{L} \rangle = \text{Tr}(\rho \boldsymbol{L})$$

Recall[5] that for any operator $\boldsymbol{A}$ and $\boldsymbol{B}$, in particular, where $\boldsymbol{AB} \neq \boldsymbol{BA}$, we still have:

$$\text{Tr}(\boldsymbol{AB}) = \text{Tr}(\boldsymbol{BA})$$

We also know[6] that, because we're dealing with *a* product state, this can't be a mixed state (it cannot be expressed as a weighted sum of multiple states), i.e if we name $|\Psi\rangle$ that (pure) product state:

$$\rho = |\Psi\rangle\langle\Psi|$$

Finally[7], again because that product state is pure, we have $\rho^2 = \rho$, which should be clear from the previous expression of ρ, as $|\Psi\rangle$ is normalized ($\sqrt{\langle\Psi|\Psi\rangle} = 1$) and the "product(s)" being associative.

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[1] The authors are a bit irregular in their use of boldface for operators; I'll try to do better, but things should be clear from the context

[2] p106, section 4.7 - *Expectation values*

[3] The trace is invariant *only* under cyclic permutations: we can't jump from $\text{Tr}(\rho^2 \boldsymbol{AB})$ to $\text{Tr}(\rho \boldsymbol{A} \rho \boldsymbol{B})$ knowing only $\text{Tr}(\boldsymbol{AB}) = \text{Tr}(\boldsymbol{BA})$.

[4] p206, section 7.5 - *Entanglement for two spins*

[5] p209, section 7.5 - *Entanglement for two spins*

[6] p202, section 7.5 - *Entanglement for two spins*

[7] p207, section 7.5 - *Entanglement for two spins*

It follows that:

$$\begin{aligned}
C(A,B) := & \ \langle \boldsymbol{AB} \rangle - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \\
= & \ Tr(\rho \boldsymbol{AB}) - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \\
= & \ Tr(\rho^2 \boldsymbol{AB}) - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \\
= & \ Tr(\rho(\boldsymbol{A}\rho\boldsymbol{B})) - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \\
= & \ \langle \boldsymbol{A}\rho\boldsymbol{B} \rangle - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \\
= & \ \langle \Psi | \boldsymbol{A}\rho\boldsymbol{B} | \Psi \rangle - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \\
= & \ \langle \Psi | \boldsymbol{A}\rho\boldsymbol{B} | \Psi \rangle - \langle \Psi | \boldsymbol{A} \underbrace{|\Psi\rangle\langle\Psi|}_{\rho} \boldsymbol{B} | \Psi \rangle \\
= & \ \boxed{0} \quad \square
\end{aligned}$$

Here's a second solution, rephrased from Michel Rennes's approach.

We start by expressing the expectation value in terms of an inner-product again, assuming we start in the state $|\Psi\rangle$:

$$\langle \boldsymbol{AB} \rangle = \langle \Psi | \boldsymbol{AB} | \Psi \rangle$$

Then, recall that $\boldsymbol{A}$ and $\boldsymbol{B}$ are two observables respectively from Alice and Bob' state spaces, which have been extended, as previously studied, so as to be able to act on a state vector $|\Psi\rangle$, taken from the composite system S_{AB}.

We definitely need this to be able to express the correlation $C(\boldsymbol{A}, \boldsymbol{B})$ in terms of those inner-products, for otherwise, the second terms in the equation below applying $\boldsymbol{A}$ or $\boldsymbol{B}$ to $|\Psi\rangle$ wouldn't make any sense:

$$C(\boldsymbol{A}, \boldsymbol{B}) = \langle \Psi | \boldsymbol{AB} | \Psi \rangle - \langle \Psi | \boldsymbol{A} | \Psi \rangle \langle \Psi | \boldsymbol{B} | \Psi \rangle$$

Hence there's an abuse of notation: with $\boldsymbol{I}_X$ being the identity operator on the space S_X:

$$\boldsymbol{A} \text{ "} = \text{" } \boldsymbol{A} \otimes \boldsymbol{I}_B; \quad \boldsymbol{B} \text{ "} = \text{" } \boldsymbol{I}_A \otimes \boldsymbol{B}$$

For clarity, I'll note $\boldsymbol{A}_A$ the observable $\boldsymbol{A}$ expressed in the system S_A, and similarly for $\boldsymbol{B}_B$:

$$\boldsymbol{A} = \boldsymbol{A}_A \otimes \boldsymbol{I}_B; \quad \boldsymbol{B} = \boldsymbol{I}_A \otimes \boldsymbol{B}_B$$

Regarding $|\Psi\rangle$, this is a product state, and we know[8] that it can be expressed as a tensor product of a state in S_A and of a state in S_B:

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$$

We can then rewrite:

$$\begin{aligned}
\langle \boldsymbol{AB} \rangle = & \ \langle \Psi | \boldsymbol{AB} | \Psi \rangle \\
= & \ (\langle \psi | \otimes \langle \phi |) \, \boldsymbol{AB} \, (|\psi\rangle \otimes |\phi\rangle) \\
= & \ (\langle \psi | \otimes \langle \phi |) \, \boldsymbol{A} \, ((\boldsymbol{I}_A \otimes \boldsymbol{B}_B) \, (|\psi\rangle \otimes |\phi\rangle)) \\
= & \ (\langle \psi | \otimes \langle \phi |) \, \boldsymbol{A} \left(\underbrace{\boldsymbol{I}_A |\psi\rangle}_{|\psi\rangle} \otimes \boldsymbol{B}_B |\phi\rangle \right) \\
= & \ (\langle \psi | \otimes \langle \phi |) \, (\boldsymbol{A}_A |\psi\rangle \otimes \boldsymbol{B}_B |\phi\rangle)
\end{aligned}$$

Where I've skipped the development for the application of $\boldsymbol{A}$ (same procedure as for applying $\boldsymbol{B}$). Then, observe[9] that $\langle \psi |$ is an operator defined on S_A, and similarly for $\langle \phi |$ being an operator defined on S_B. Their tensor product is then an operator defined on S_{AB} and the usual rules for applying this combined operator hold:

$$\begin{aligned}
\langle \boldsymbol{AB} \rangle = & \ (\langle \psi | \boldsymbol{A}_A | \psi \rangle) \otimes (\langle \phi | \boldsymbol{B}_B | \phi \rangle) \\
= & \ \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle
\end{aligned}$$

[8] p164, section 6.5 - *Product states*

[9] It would be interesting to formalized that more thoroughly. If I'm not mistaken the idea is that the bras of $S_A \otimes S_B$ can be expressed as a combination of one bra from S_A and one bra from S_B. More precisely, the bras being elements of the dual spaces, it's because of the following (canonical) isomorphism: $S_{AB}^* = (S_A \otimes S_B)^* \cong S_A^* \otimes S_B^*$, see for instance `https://planetmath.org/tensorproductofdualspacesisadualspaceoftensorproduct`

Hence clearly, $\boxed{C(\boldsymbol{A},\boldsymbol{B}) := \langle \boldsymbol{AB}\rangle - \langle \boldsymbol{A}\rangle \langle \boldsymbol{B}\rangle = 0}$.

For completeness, here's one last solution, rephrased from Filip Van Lijsebetten's approach (p52), which relies on the probabilistic definition of the average value.

Remember that the average value of an observable $\boldsymbol{L}$ is (mathematically) defined[10] as:

$$\langle \boldsymbol{L}\rangle := \sum_i \lambda_i P(\lambda_i)$$

Hence:

$$\langle \boldsymbol{AB}\rangle = \sum_{ab} \lambda_{ab} P(\lambda_{ab}); \quad \langle \boldsymbol{A}\rangle = \sum_{a} \lambda_{a} P(\lambda_{a}); \quad \langle \boldsymbol{B}\rangle = \sum_{b} \lambda_{b} P(\lambda_{b})$$

Recall that the ab corresponds to all labels created by concatenating all potential values for a and b. This means that we'll have $\sum_{ab} = \sum_{a,b} := \sum_a \sum_b$. Let's rewrite the correlation $C(\boldsymbol{A},\boldsymbol{B})$:

$$\begin{aligned}
C(\boldsymbol{A},\boldsymbol{B}) := &\quad \langle \boldsymbol{AB}\rangle - \langle \boldsymbol{A}\rangle \langle \boldsymbol{B}\rangle \\
= &\quad \left(\sum_{ab} \lambda_{ab} P(\lambda_{ab})\right) - \left(\sum_{a} \lambda_{a} P(\lambda_{a})\right)\left(\sum_{b} \lambda_{b} P(\lambda_{b})\right) \\
= &\quad \left(\sum_{ab} \lambda_{ab} P(\lambda_{ab})\right) - \left(\sum_{a} \lambda_{a} P(\lambda_{a})\left(\sum_{b} \lambda_{b} P(\lambda_{b})\right)\right) \\
= &\quad \left(\sum_{ab} \lambda_{ab} P(\lambda_{ab})\right) - \left(\sum_{a}\sum_{b} \lambda_{a} P(\lambda_{a})\lambda_{b} P(\lambda_{b})\right) \\
= &\quad \left(\sum_{ab} \lambda_{ab} P(\lambda_{ab})\right) - \left(\sum_{a,b} \lambda_{a}\lambda_{b} P(\lambda_{a}) P(\lambda_{b})\right) \\
= &\quad \sum_{a,b}\Big(\lambda_{ab} P(\lambda_{ab}) - \lambda_{a}\lambda_{b} P(\lambda_{a}) P(\lambda_{b})\Big)
\end{aligned}$$

Now the notation is a bit confusing[11], but recall than λ_{ab} corresponds to the value we get for our combined state (which occurs with a probability of $P(\lambda_{ab})$). And this precisely corresponds the fact that we have λ_a in the subspace S_A and λ_b in the subspace S_B: so we can read it like $\lambda_{ab} \simeq \lambda_a\lambda_b$. Hence this factors as:

$$C(\boldsymbol{A},\boldsymbol{B}) = \sum_{a,b} \lambda_{ab}\big(P(\lambda_{ab}) - P(\lambda_a)P(\lambda_b)\big)$$

Remark 2. *So far, we've essentially just restated with a different notation what we did in L06E01*

Now by definition for a product state, there is independence between the two "events": the measurement of either A or B doesn't affect the other one. That is, $P(\lambda_{ab}) = P(\lambda_a)P(\lambda_b)$[12], hence the correlation really is zero. □

[10] p105, section 4.7 - *Expectation values*

[11] I could have made things a bit clearer: for instance, we really have three different probability distributions, one for each state involved, but they are all denoted very similarly.

[12] This is the definition of independence of events in ordinary probability theory: `https://en.wikipedia.org/wiki/Independence_(probability_theory)#For_events`