The Theoretical Minimum

Quantum Mechanics - Solutions

L07E10

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Exercise 1. *Verify that the state-vector in* 7.30 *represents a completely untangled state.*

Let's recall the state-vector from 7.30, and let's call it $|\Psi\rangle$.

$$|\Psi\rangle = \alpha_u|u, b\rangle + \alpha_d|d, b\rangle$$

As I've found this confusing, let me start by recalling a bit of vocabulary[1]. A quantum state can be either **pure** or **mixed**: either its a single state, or a *convex combination*[2] of pure states. This is true for a "regular" state space, as for a state space built via a tensor products of two (or finitely many, by induction) other state spaces.

Now there's a second qualification, that is only applicable for states which are taken from a state space made by glueing two (again, or finitely many) other state spaces: **entangled** states, and **disentangled** states.

Mixed and *entangled* are definitely not synonymous: you can have a non-mixed (i.e. pure) entangled state for example.

Example 1. *The state vector from* 7.30 *is a pure state: this is not a convex combination of states. But this tells us absolutely nothing regarding whether it's an entangled state. We know however that it makes sense to talk about it being entangled or not, as we're dealing with a combined system involving* (i) *an apparatus and* (ii) *a spin to be measured by said apparatus.*

We could test this purity by computing the density matrix ρ*, and checking whether* $\rho^2 = \rho$ *or* $Tr(\rho) = 1$.

Let's clarify the vocabulary one step further: a **completely untangled state** *is* a **product state**: that's a state where measurements on one subsystem affect in no ways the other subsystem(s).

From there, we have a few different ways of proceeding.

The simplest approach is to remember that a state is a product state when it can be expressed via two components (well, or more, but we're in the case where there are two subsystems here: the apparatus, and the spin to be measured with the apparatus), one for each subsystem. Recall that $|a, \alpha\rangle$ really is a shortcut for $|a\rangle \otimes |\alpha\}$. This means, the prepared state really is:

$$\alpha_u|u\rangle \otimes |b\} + \alpha_d|d\rangle \otimes |b\}$$

[1] See for instance: `https://www.researchgate.net/post/What-is-difference-between-mixed-state-and-mixed-entangled-state`

[2] A fancy term you may find here and there: a linear combination of elements, where the scalars factors sums to 1; see `https://en.wikipedia.org/wiki/Convex_combination`

But the tensor product distributes[3], hence this simplifies as:

$$(\alpha_u|u\rangle + \alpha_d|d\rangle) \otimes |b\}$$

As the combined state is normalized, we must have $\sqrt{\alpha_u^2 + \alpha_d^2} = 1$, which implies that the sub-state corresponding to the spin is also normalized. Trivially, the sub-state corresponding to the apparatus is also normalized. Hence, we've expressed our combined state as a tensor product of two normalized state, one for each subsystems: $\boxed{\text{this is a product state}}$.

A slightly more involved (calculus-wise) variant of this approach would be to rely on the general form of the product state[4] and to evaluate whether our state vector can be expressed in such a way. The general form can be computed, again using the distributive nature of the tensor product:

$$\begin{aligned}
|\text{product state}\rangle &= \left\{\alpha_u|u\rangle + \alpha_d|d\rangle\right\} \otimes \left\{\beta_b|b\} + \beta_+|+1\} + \beta_-|-1\}\right\} \\
&= \alpha_u|u\rangle \otimes \left\{\beta_b|b\} + \beta_+|+1\} + \beta_-|-1\}\right\} + \alpha_d|d\rangle\left\{\beta_b|b\} + \beta_+|+1\} + \beta_-|-1\}\right\} \\
&= \alpha_u\beta_d|u,b\rangle + \alpha_u\beta_+|u,+1\rangle + \alpha_u\beta_-|u,-1\rangle + \alpha_d\beta_d|d,b\rangle + \alpha_d\beta_+|d,+1\rangle + \alpha_d\beta_-|d,-1\rangle
\end{aligned}$$

By setting:

$$\beta_d = 1; \beta_+ = \beta_- = 0$$

We found back the state vector from 7.30, retrospectively justifying the notation for α_u and α_d. Because the subsystem states, must be normalized, the resulting combined state is also normalized.

However, and perhaps this is more in line with the author's intent, we've just saw[5] two tests to check whether the state corresponding to a given a wave-function for a composite system is entangled or not.

For the first criteria, we'd need to take any two arbitrary observables from each subsystem, say observable $\boldsymbol{A}$ and $\boldsymbol{B}$, and prove that their correlation $C(\boldsymbol{A}, \boldsymbol{B})$ is zero. But essentially, the proof will end up relying on the tensor product distributivity, rely on the density matrix (see just after), or essentially mimick the proofs of L07E09.

I don't think there's added value to develop it further here.

The second technique is slightly more original: the idea is that, for any product state, the density matrix has exactly one non-zero eigenvalue, and that eigenvalue is exactly 1.

Recall that in L07E04 we've already determined Alice's density matrix:

$$\rho = \begin{pmatrix} \alpha_u^*\alpha_u & \alpha_u^*\alpha_d \\ \alpha_d^*\alpha_u & \alpha_d^*\alpha_d \end{pmatrix}$$

Let's diagonalize it: as usual, we have the eigenvector/eigenvalue relationship:

$$\rho|\lambda\rangle = \lambda|\lambda\rangle \Leftrightarrow (\rho - I_2\lambda)|\lambda\rangle = 0$$

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[3]As is common in most Physics-centered introduction to Quantum Mechanics, the tensor product introduction is a bit hand-wavy. For a more rigorous development, see for instance this video by F. Schuller. Some subtleties such as the fact that the equivalence classes respect addition and scalar multiplication have been left as homework; there's a set of notes which contains the "missing" proofs.

[4]p164, section 6.5 - *Product states*

[5]p212 and onward, section 7.7 *Tests for Entanglement*

Which implies that $\rho - I_2\lambda$ isn't invertible[6], which translates to its determinant being equal to zero:

$$\begin{aligned}
\begin{vmatrix} \alpha_u^*\alpha_u - \lambda & \alpha_u^*\alpha_d \\ \alpha_d^*\alpha_u & \alpha_d^*\alpha_d - \lambda \end{vmatrix} &= \Big((\alpha_u^*\alpha_u - \lambda)(\alpha_d^*\alpha_d - \lambda)\Big) - \Big(\alpha_u^*\alpha_d\alpha_d^*\alpha_u\Big) \\
&= \Big(\alpha_u^*\alpha_u\alpha_d^*\alpha_d - \lambda(\underbrace{\alpha_u^*\alpha_u + \alpha_d^*\alpha_d}_{=\langle\Psi|\Psi\rangle=1}) + \lambda^2\Big) - \alpha_u^*\alpha_d\alpha_d^*\alpha_u \\
&= \lambda(1-\lambda)
\end{aligned}$$

Clearly, we have one non-zero eigenvalue which is exactly one: the criteria indeed applies, and the state must be non-entangled.

[6] Multiply both side of the equation by the inverse, and use the fact that an eigenvector cannot be the zero vector