The Theoretical Minimum

Quantum Mechanics - Solutions

L07E12

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Verify the numerical values in each rap sheet.*

Remark 1. *This is a long "solution": I'm taking the time to rederive some results that were previously established either in the course, or in earlier exercises, while, clarifying what I think is a important source of confusion: the elements of an Hilbert space (the so-called "state-space") aren't the states of a quantum system. More on that when exploring the singlet state.*

Some numerical results have been automatically computed by a R script, inlined at the end of this exercise.

Let's start by recalling a few things before diving in.

We are, as usual, in the case of two spin systems, one for Alice, S_A, one for Bob, S_B. A composite system S_{AB} is then created from those two via a tensor product.

When relevant, we'll be using the usual ordered bases:

- $\{|u\rangle, |d\rangle\}$ for S_A;
- $\{|u\}, |d\}\}$ for S_B;
- and $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$ for S_{AB}.

Then, let's recall the Pauli matrices corresponding to the observables associated to the spin "components": (from Eqs. 3.20, at the end of section 3.4)

$$\boldsymbol{\tau_z}^B = \boldsymbol{\sigma_z}^A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad \boldsymbol{\tau_x}^B = \boldsymbol{\sigma_x}^A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \boldsymbol{\tau_y}^B = \boldsymbol{\sigma_y}^A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Implicitly, the matrices are expressed in the ordered basis of the corresponding sub-system, S_A for $\boldsymbol{\sigma_i}$[1] and S_B for $\boldsymbol{\tau_i}^B$[1]. We will also need the matrix forms for the lifted operators from the sub-systems to the composite system, where $\boldsymbol{I}^A$ is the identity operator on S_A and $\boldsymbol{I}^B$ the identity operator on S_B, again implicitly relying on S_{AB}'s usual ordered basis.

Remark 2. *We could have used the equivalent tables provided in annexes in the book, or the results of previous exercises.*

Remark 3. *Note that the final result appears twice below: this isn't an error. The first occurence is a manual computation, while the second has been automatically performed in R.*

[1] Strictly speaking, the operators aren't equal; their matrix representation in their respective-basis are. The equals signs are to be understood in this context.

$$\boldsymbol{\sigma_z} := \boldsymbol{\sigma_z^A} \otimes \boldsymbol{I}^B = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$\boldsymbol{\tau_z} := \boldsymbol{I}^A \otimes \boldsymbol{\tau_z^B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$\boldsymbol{\sigma_x} := \boldsymbol{\sigma_x^A} \otimes \boldsymbol{I}^B = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

$$\boldsymbol{\tau_x} := \boldsymbol{I}^A \otimes \boldsymbol{\tau_x^B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$\boldsymbol{\sigma_y} := \boldsymbol{\sigma_y^A} \otimes \boldsymbol{I}^B = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\\ i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\\ i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix}$$

$$\boldsymbol{\tau_y} := \boldsymbol{I}^A \otimes \boldsymbol{\tau_y^B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix} = \begin{pmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix}$$

We'll also need a few more "combined" observables. I'll skip the manual matrix multiplication here: those have been automatically computed by R:

$$\boldsymbol{\sigma_z \tau_z} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}; \qquad \boldsymbol{\tau_z \sigma_z} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\boldsymbol{\tau_x \sigma_x} = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}; \qquad \boldsymbol{\tau_y \sigma_y} = \begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}$$

Finally, let's recall one more time the formula for the expectation value $\langle \boldsymbol{L} \rangle$ of an observable $\boldsymbol{L}$ of a system in a state $|\Psi\rangle$:

$$\langle \boldsymbol{L} \rangle = \langle \Psi | \boldsymbol{L} | \Psi \rangle$$

Product state

We're starting from the following state-vector for the composite system, and using various properties of the tensor product of vector states we progressively reach:

$$\begin{aligned}
|\Psi\rangle &= \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle \\
&= \alpha_u\beta_u|u\rangle \otimes |u\} + \alpha_u\beta_d|u\rangle \otimes |d\} + \alpha_d\beta_u|d\rangle \otimes |u\} + \alpha_d\beta_d|d\rangle \otimes |d\} \\
&= (\alpha_u|u\rangle) \otimes (\beta_u|u\}) + (\alpha_u|u\rangle) \otimes (\beta_d|d\}) + (\alpha_d|d\rangle) \otimes (\beta_u|u\}) + (\alpha_d|d\rangle) \otimes (\beta_d|d\}) \\
&= \alpha_u|u\rangle \otimes (\beta_d|d\} + \beta_u|u\}) + \alpha_d|d\rangle \otimes (\beta_u|u\} + \beta_d|d\}) \\
&= \underbrace{(\alpha_u|u\rangle + \alpha_d|d\rangle)}_{=:|\phi\rangle} \otimes \underbrace{(\beta_d|d\} + \beta_u|u\})}_{=:|\psi\}}
\end{aligned}$$

We've verified that this particular composite state is a state product: it can be expressed as the tensor product of two states, $|\phi\rangle \in S_A$ and $|\psi\} \in S_B$.

The normalization condition yields:

$$\begin{aligned}
\||\Psi\rangle\| := \sqrt{\langle\Psi|\Psi\rangle} = 1 \Leftrightarrow \quad & \sqrt{\alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d + \alpha_d^*\beta_u^*\alpha_d\beta_u + \alpha_d^*\beta_d^*\alpha_d\beta_d} = 1 \\
\Leftrightarrow \quad & \sqrt{\alpha_u^*\alpha_u(\beta_u^*\beta_u + \beta_d^*\beta_d) + \alpha_d^*\alpha_d(\beta_u^*\beta_u + \beta_d^*\beta_d)} = 1 \\
\Leftrightarrow \quad & \sqrt{(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d)} = 1 \\
\Leftrightarrow \quad & (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d) = 1 & (\forall\gamma \in \mathbb{C}),\ \gamma^*\gamma =: |\gamma| > 0 \\
\Leftrightarrow \quad & \begin{cases} \alpha_u^*\alpha_u + \alpha_d^*\alpha_d & = 1 \\ \beta_u^*\beta_u + \beta_d^*\beta_d & = 1 \end{cases} \\
\Leftrightarrow \quad & \begin{cases} \sqrt{\alpha_u^*\alpha_u + \alpha_d^*\alpha_d} & = 1 \\ \sqrt{\beta_u^*\beta_u + \beta_d^*\beta_d} & = 1 \end{cases} & (\forall\gamma \in \mathbb{C}),\ \gamma^*\gamma =: |\gamma| > 0 \\
\Leftrightarrow \quad & \begin{cases} \||\phi\rangle\| & = 1 \\ \||\psi\}\| & = 1 \end{cases}
\end{aligned}$$

We've verified consistency of the normalization condition between the composite system, and the sub-systems: our composite state vector is normalized iff the individual vectors from the sub-systems are normalized.

Moving on to the density matrices, let's do the reasoning for Alice's state only, as the same argument applies to Bob's. Let's first stay in S_A. We know that the subsystem's state is a *pure* state $|\phi\rangle$: it's a convex combination[2] with a single term. The density matrix ρ^A is thus:

$$\rho^A = 1|\phi\rangle\langle\phi| = \begin{pmatrix} \alpha_u \\ \alpha_d \end{pmatrix} \begin{pmatrix} \alpha_u^* & \alpha_d^* \end{pmatrix} = \begin{pmatrix} \alpha_u\alpha_u^* & \alpha_u\alpha_d^* \\ \alpha_d\alpha_u^* & \alpha_d\alpha_d^* \end{pmatrix}$$

It is immediate to check that ρ^A is Hermitian ($\rho^A = (\rho^A)^\dagger$, as $(\alpha_u\alpha_d^*)^* = \alpha_d\alpha_u^*$). Furthermore:

$$\text{Tr}(\rho^A) = \alpha_u\alpha_u^* + \alpha_d\alpha_d^* = 1 \text{ (because of the normalization condition)}$$

We also have:

$$\begin{aligned}
(\rho^A)^2 &= \begin{pmatrix} \alpha_u\alpha_u^* & \alpha_u\alpha_d^* \\ \alpha_d\alpha_u^* & \alpha_d\alpha_d^* \end{pmatrix} \begin{pmatrix} \alpha_u\alpha_u^* & \alpha_u\alpha_d^* \\ \alpha_d\alpha_u^* & \alpha_d\alpha_d^* \end{pmatrix} \\
&= \begin{pmatrix} (\alpha_u\alpha_u^*)(\alpha_u\alpha_u^*) + (\alpha_u\alpha_d^*)(\alpha_d\alpha_u^*) & (\alpha_u\alpha_u^*)(\alpha_u\alpha_d^*) + (\alpha_u\alpha_d^*)(\alpha_d\alpha_d^*) \\ (\alpha_d\alpha_u^*)(\alpha_u\alpha_u^*) + (\alpha_d\alpha_d^*)(\alpha_d\alpha_u^*) & (\alpha_d\alpha_u^*)(\alpha_u\alpha_d^*) + (\alpha_d\alpha_d^*)(\alpha_d\alpha_d^*) \end{pmatrix} \\
&= \begin{pmatrix} (\alpha_u\alpha_u^*)\underbrace{(\alpha_u\alpha_u^* + \alpha_d^*\alpha_d)}_{=1} & (\alpha_u\alpha_d^*)\underbrace{(\alpha_u\alpha_u^* + \alpha_d^*\alpha_d)}_{=1} \\ (\alpha_d\alpha_u^*)\underbrace{(\alpha_u\alpha_u^* + \alpha_d^*\alpha_d)}_{=1} & (\alpha_d\alpha_d^*)\underbrace{(\alpha_u\alpha_u^* + \alpha_d^*\alpha_d)}_{=1} \end{pmatrix} \\
&= \rho^A
\end{aligned}$$

And naturally, $\text{Tr}((\rho^A)^2) = \text{Tr}(\rho) = 1$. Those last two conditions are indeed we expect for a pure state.

Let's move on to diagonalizing ρ^A. As usual, eigenvectors $|\lambda\rangle$ are tied to their corresponding eigenvalues

[2] https://en.wikipedia.org/wiki/Convex_combination

λ via:

$$\begin{aligned}
\rho^A|\lambda\rangle = \lambda|\lambda\rangle \Leftrightarrow (\rho^A - \boldsymbol{I}^A\lambda)|\lambda\rangle = 0 \Leftrightarrow \quad & |\rho^A - \boldsymbol{I}^A\lambda| = 0 \\
\Leftrightarrow \quad & \begin{vmatrix} \alpha_u\alpha_u^* - \lambda & \alpha_u\alpha_d^* \\ \alpha_d\alpha_u^* & \alpha_d\alpha_d^* - \lambda \end{vmatrix} \\
\Leftrightarrow \quad & (\alpha_u\alpha_u^* - \lambda)(\alpha_d\alpha_d^* - \lambda) - \alpha_d\alpha_u^*\alpha_u\alpha_d^* = 0 \\
\Leftrightarrow \quad & \lambda^2 - \underbrace{(\alpha_u\alpha_u^* + \alpha_d\alpha_d^*)}_{=1}\lambda = 0 \\
\Leftrightarrow \quad & \lambda(\lambda - 1) = 0 \\
\Leftrightarrow \quad & \begin{cases} \lambda & = 0 \\ \lambda & = 1 \end{cases}
\end{aligned}$$

Let's verify that the eigenvector $|\lambda_1\rangle$ associated to $\lambda = 1$ is indeed the wave-function associated to Alice's sub-system, i.e with components α_u and α_d

$$\rho^A|\lambda_1\rangle = \begin{pmatrix} \alpha_u\alpha_u^* & \alpha_u\alpha_d^* \\ \alpha_d\alpha_u^* & \alpha_d\alpha_d^* \end{pmatrix} \begin{pmatrix} \alpha_u \\ \beta_d \end{pmatrix} = \begin{pmatrix} \alpha_u\alpha_u^*\alpha_u + \alpha_u\alpha_d^*\alpha_d \\ \alpha_d\alpha_u^*\alpha_u + \alpha_d\alpha_d^*\alpha_d \end{pmatrix} = \begin{pmatrix} \alpha_u\underbrace{(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)}_{=1} \\ \alpha_d\underbrace{(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)}_{=1} \end{pmatrix} = 1|\lambda_1\rangle$$

Again, by symmetry, we immediately have for S_B:

$$\rho_B = \begin{pmatrix} \beta_u\beta_u^* & \beta_u\beta_d^* \\ \beta_d\beta_u^* & \beta_d\beta_d^* \end{pmatrix}; \qquad \rho_B^2 = \rho_B; \qquad \mathrm{Tr}(\rho_B^2) = \mathrm{Tr}(\rho_B) = 1$$

Same eigenvalues/eigenvectors.

What about S_{AB}? Well, $|\Psi\rangle$ still is a pure state in S_{AB}, meaning, its density matrix again is a convex combination involving a single term; expanding it as a matrix in S_{AB}'s usual ordered basis yields:

$$\begin{aligned}
\rho = 1|\Psi\rangle\langle\Psi| = \quad & \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \begin{pmatrix} (\alpha_u\beta_u)^* & (\alpha_u\beta_d)^* & (\alpha_d\beta_u)^* + (\alpha_d\beta_d)^* \end{pmatrix} \\
= \quad & \begin{pmatrix} \alpha_u\beta_u\alpha_u^*\beta_u^* & \alpha_u\beta_u\alpha_u^*\beta_d^* & \alpha_u\beta_u\alpha_d^*\beta_u^* & \alpha_u\beta_u\alpha_d^*\beta_d^* \\ \alpha_u\beta_d\alpha_u^*\beta_u^* & \alpha_u\beta_d\alpha_u^*\beta_d^* & \alpha_u\beta_d\alpha_d^*\beta_u^* & \alpha_u\beta_d\alpha_d^*\beta_d^* \\ \alpha_d\beta_u\alpha_u^*\beta_u^* & \alpha_d\beta_u\alpha_u^*\beta_d^* & \alpha_d\beta_u\alpha_d^*\beta_u^* & \alpha_d\beta_u\alpha_d^*\beta_d^* \\ \alpha_d\beta_d\alpha_u^*\beta_u^* & \alpha_d\beta_d\alpha_u^*\beta_d^* & \alpha_d\beta_d\alpha_d^*\beta_u^* & \alpha_d\beta_d\alpha_d^*\beta_d^* \end{pmatrix}
\end{aligned}$$

Because this is again a pure state, this time in the S_{AB} system, we expect the usual formulas to hold. Let's check them for good measure:

$$\begin{aligned}
\mathrm{Tr}(\rho) = \quad & \alpha_u\beta_u\alpha_u^*\beta_u^* + \alpha_u\beta_d\alpha_u^*\beta_d^* + \alpha_d\beta_u\alpha_d^*\beta_u^* + \alpha_d\beta_d\alpha_d^*\beta_d^* \\
= \quad & \alpha_u\alpha_u^*\underbrace{(\beta_u\beta_u^* + \beta_d\beta_d^*)}_{=1} + \alpha_d\alpha_d^*\underbrace{(\beta_u\beta_u^* + \beta_d\beta_d^*)}_{=1} \\
= \quad & 1
\end{aligned}$$

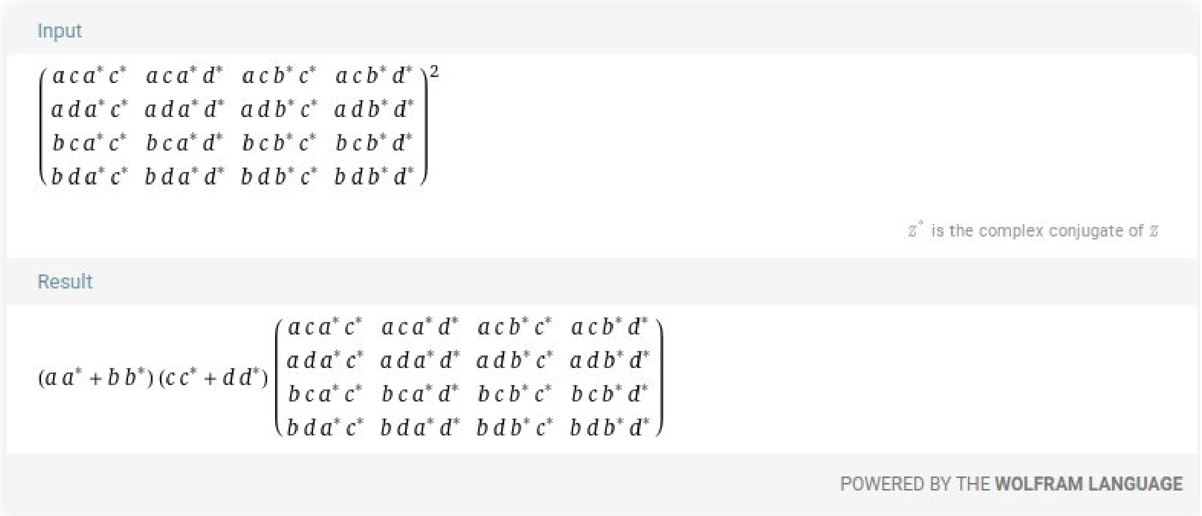
The following is going to be too tedious to do by hand:

$$\begin{aligned}
\rho^2 = & \begin{pmatrix} \alpha_u\beta_u\alpha_u^*\beta_u^* & \alpha_u\beta_u\alpha_u^*\beta_d^* & \alpha_u\beta_u\alpha_d^*\beta_u^* & \alpha_u\beta_u\alpha_d^*\beta_d^* \\ \alpha_u\beta_d\alpha_u^*\beta_u^* & \alpha_u\beta_d\alpha_u^*\beta_d^* & \alpha_u\beta_d\alpha_d^*\beta_u^* & \alpha_u\beta_d\alpha_d^*\beta_d^* \\ \alpha_d\beta_u\alpha_u^*\beta_u^* & \alpha_d\beta_u\alpha_u^*\beta_d^* & \alpha_d\beta_u\alpha_d^*\beta_u^* & \alpha_d\beta_u\alpha_d^*\beta_d^* \\ \alpha_d\beta_d\alpha_u^*\beta_u^* & \alpha_d\beta_d\alpha_u^*\beta_d^* & \alpha_d\beta_d\alpha_d^*\beta_u^* & \alpha_d\beta_d\alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u\alpha_u^*\beta_u^* & \alpha_u\beta_u\alpha_u^*\beta_d^* & \alpha_u\beta_u\alpha_d^*\beta_u^* & \alpha_u\beta_u\alpha_d^*\beta_d^* \\ \alpha_u\beta_d\alpha_u^*\beta_u^* & \alpha_u\beta_d\alpha_u^*\beta_d^* & \alpha_u\beta_d\alpha_d^*\beta_u^* & \alpha_u\beta_d\alpha_d^*\beta_d^* \\ \alpha_d\beta_u\alpha_u^*\beta_u^* & \alpha_d\beta_u\alpha_u^*\beta_d^* & \alpha_d\beta_u\alpha_d^*\beta_u^* & \alpha_d\beta_u\alpha_d^*\beta_d^* \\ \alpha_d\beta_d\alpha_u^*\beta_u^* & \alpha_d\beta_d\alpha_u^*\beta_d^* & \alpha_d\beta_d\alpha_d^*\beta_u^* & \alpha_d\beta_d\alpha_d^*\beta_d^* \end{pmatrix} \\
= \quad & \ldots \\
= \quad & \rho
\end{aligned}$$

With some text reformatting and a few tweak, we can convert the LaTeXcode in a Wolfram Alpha matrix:

```
{{a c Conjugate(a) Conjugate(c),a c Conjugate(a) Conjugate(d),
    a c Conjugate(b) Conjugate(c),
    a c Conjugate(b) Conjugate(d)},
{a d Conjugate(a) Conjugate(c),
    a d Conjugate(a) Conjugate(d),
    a d Conjugate(b) Conjugate(c),
    a d Conjugate(b) Conjugate(d)},
{b c Conjugate(a) Conjugate(c),
    b c Conjugate(a) Conjugate(d),
    b c Conjugate(b) Conjugate(c),
    b c Conjugate(b) Conjugate(d)},
{b d Conjugate(a) Conjugate(c),
    b d Conjugate(a) Conjugate(d),
    b d Conjugate(b) Conjugate(c),
    b d Conjugate(b) Conjugate(d)}}
```

And use for instance the web interface (see `https://www.wolframalpha.com/input?i2d=true&i=Power%5B%7B%7Ba+c+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Ca+c+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%2Ca+c+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Ca+c+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%7D%2C%7Ba+d+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Ca+d+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%2Ca+d+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Ca+d+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%7D%2C%7Bb+c+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Cb+c+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%2Cb+c+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Cb+c+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%7D%2C%7Bb+d+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Cb+d+Conjugate%5C%2840%29a%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%2Cb+d+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29c%5C%2841%29%2Cb+d+Conjugate%5C%2840%29b%5C%2841%29+Conjugate%5C%2840%29d%5C%2841%29%7D%7D%2C2%5D`) to make sure it works:



Input

$$\begin{pmatrix} a\,c\,a^*\,c^* & a\,c\,a^*\,d^* & a\,c\,b^*\,c^* & a\,c\,b^*\,d^* \\ a\,d\,a^*\,c^* & a\,d\,a^*\,d^* & a\,d\,b^*\,c^* & a\,d\,b^*\,d^* \\ b\,c\,a^*\,c^* & b\,c\,a^*\,d^* & b\,c\,b^*\,c^* & b\,c\,b^*\,d^* \\ b\,d\,a^*\,c^* & b\,d\,a^*\,d^* & b\,d\,b^*\,c^* & b\,d\,b^*\,d^* \end{pmatrix}^2$$

z^* is the complex conjugate of z

Result

$$(a\,a^* + b\,b^*)\,(c\,c^* + d\,d^*) \begin{pmatrix} a\,c\,a^*\,c^* & a\,c\,a^*\,d^* & a\,c\,b^*\,c^* & a\,c\,b^*\,d^* \\ a\,d\,a^*\,c^* & a\,d\,a^*\,d^* & a\,d\,b^*\,c^* & a\,d\,b^*\,d^* \\ b\,c\,a^*\,c^* & b\,c\,a^*\,d^* & b\,c\,b^*\,c^* & b\,c\,b^*\,d^* \\ b\,d\,a^*\,c^* & b\,d\,a^*\,d^* & b\,d\,b^*\,c^* & b\,d\,b^*\,d^* \end{pmatrix}$$

POWERED BY THE **WOLFRAM LANGUAGE**

Remark 4. *The leading factors correspond to the norms of the sub-system states: it's a* 1 *in disguise.*

It follows that $\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1$.

We've already brushed upon it, but let's make things crystal clear regarding the wave-functions: we have one wave function for each sub-systems:

$$\begin{aligned} \psi^A(|u\rangle) = {} & \alpha_u; \qquad \psi^B(|u\}) = \beta_u; \\ \psi^A(|d\rangle) = {} & \alpha_d; \qquad \psi^B(|d\}) = \beta_d. \end{aligned}$$

And a wave-function for the composite system, which indeed factorize as a product of the sub-systems wave-functions:

$$\psi : \begin{cases} |uu\rangle & \to \alpha_u\beta_u = \psi^A(|u\rangle)\psi^B(|u\}) \\ |ud\rangle & \to \alpha_u\beta_d = \psi^A(|u\rangle)\psi^B(|d\}) \\ |du\rangle & \to \alpha_d\beta_u = \psi^A(|d\rangle)\psi^B(|u\}) \\ |dd\rangle & \to \alpha_d\beta_d = \psi^A(|d\rangle)\psi^B(|d\}) \end{cases} \Leftrightarrow \psi(a,b) = \psi^A(a)\psi^B(b)$$

Finally, we can crunch some numbers using the usual expectation value formula, using the matrices for the spin observables we've re-established earlier. This could have been automated, but I haven't looked much into how to perform symbolic computation with R.

$$\begin{aligned}
\langle\boldsymbol{\sigma_x}\rangle = \langle\Psi|\boldsymbol{\sigma_x}|\Psi\rangle = &\quad \begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* & \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix}\alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d\end{pmatrix} \\
= &\quad \begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* & \end{pmatrix} \begin{pmatrix}\alpha_d\beta_u \\ \alpha_d\beta_d \\ \alpha_u\beta_u \\ \alpha_u\beta_d\end{pmatrix} \\
= &\quad \alpha_u^*\beta_u^*\alpha_d\beta_u + \alpha_u^*\beta_d^*\alpha_d\beta_d + \alpha_d^*\beta_u^*\alpha_u\beta_u + \alpha_d^*\beta_d^*\alpha_u\beta_d \\
= &\quad \beta_u^*\beta_u(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) + \beta_d^*\beta_d(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) \\
= &\quad (\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)\underbrace{(\beta_u^*\beta_u + \beta_d^*\beta_d)}_{=1} \\
= &\quad \alpha_u^*\alpha_d + \alpha_d^*\alpha_u \\
\Rightarrow \langle\boldsymbol{\sigma_x}\rangle^2 = &\quad (\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)^2 \\
= &\quad (\alpha_u^*\alpha_d)^2 + 2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + (\alpha_d^*\alpha_u)^2
\end{aligned}$$

$$\begin{aligned}
\langle\boldsymbol{\sigma_y}\rangle = \langle\Psi|\boldsymbol{\sigma_y}|\Psi\rangle = &\quad \begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* & \end{pmatrix} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix}\alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d\end{pmatrix} \\
= &\quad i\begin{pmatrix}\alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* & \end{pmatrix} \begin{pmatrix}-\alpha_d\beta_u \\ -\alpha_d\beta_d \\ \alpha_u\beta_u \\ \alpha_u\beta_d\end{pmatrix} \\
= &\quad i(-\alpha_u^*\beta_u^*\alpha_d\beta_u - \alpha_u^*\beta_d^*\alpha_d\beta_d + \alpha_d^*\beta_u^*\alpha_u\beta_u + \alpha_d^*\beta_d^*\alpha_u\beta_d) \\
= &\quad i(-\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)\underbrace{(\beta_u^*\beta_u + \beta_d^*\beta_d)}_{=1} \\
= &\quad i(-\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) \\
\Rightarrow \langle\boldsymbol{\sigma_y}\rangle^2 = &\quad (i(-\alpha_u^*\alpha_d + \alpha_d^*\alpha_u))^2 \\
= &\quad -(\alpha_d^*\alpha_u)^2 + 2\alpha_d^*\alpha_u\alpha_u^*\alpha_d - (\alpha_u^*\alpha_d)^2
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{\sigma_z} \rangle = \langle \Psi | \boldsymbol{\sigma_z} | \Psi \rangle = &\begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \\
= &\begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ -\alpha_d\beta_u \\ -\alpha_d\beta_d \end{pmatrix} \\
= &\ \alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d - \alpha_d^*\beta_u^*\alpha_d\beta_u - \alpha_d^*\beta_d^*\alpha_d\beta_d \\
= &\ \beta_u^*\beta_u(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d) + \beta_d^*\beta_d(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d) \\
= &\ (\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)\underbrace{(\beta_u^*\beta_u + \beta_d^*\beta_d)}_{=1} \\
= &\ \alpha_u^*\alpha_u - \alpha_d^*\alpha_d \\
\Rightarrow \langle \boldsymbol{\sigma_z} \rangle^2 = &\ (\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)^2 \\
= &\ (\alpha_u^*\alpha_u)^2 - 2\alpha_u^*\alpha_u\alpha_d^*\alpha_d + (\alpha_d^*\alpha_d)^2
\end{aligned}$$

We can now compute:

$$\begin{aligned}
\langle \boldsymbol{\sigma_x} \rangle^2 + \langle \boldsymbol{\sigma_y} \rangle^2 + \langle \boldsymbol{\sigma_z} \rangle^2 = &\ (\alpha_u^*\alpha_d)^2 + 2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + (\alpha_d^*\alpha_u)^2 \\
&- (\alpha_d^*\alpha_u)^2 + 2\alpha_d^*\alpha_u\alpha_u^*\alpha_d - (\alpha_u^*\alpha_d)^2 \\
&+ (\alpha_u^*\alpha_u)^2 - 2\alpha_u^*\alpha_u\alpha_d^*\alpha_d + (\alpha_d^*\alpha_d)^2 \\
= &\ (\alpha_u^*\alpha_u)^2 + 2\alpha_u^*\alpha_u\alpha_d^*\alpha_d + (\alpha_d^*\alpha_d)^2 \\
= &\ (\underbrace{\alpha_u^*\alpha_u + \alpha_d^*\alpha_d}_{=1})^2 \\
= &\ \boxed{1}
\end{aligned}$$

Moving on to the other spin "components":

$$\begin{aligned}
\langle \boldsymbol{\tau_x} \rangle = \langle \Psi | \boldsymbol{\tau_x} | \Psi \rangle = &\begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \\
= &\begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} \alpha_u\beta_d \\ \alpha_u\beta_u \\ \alpha_d\beta_d \\ \alpha_d\beta_u \end{pmatrix} \\
= &\ \alpha_u^*\beta_u^*\alpha_u\beta_d + \alpha_u^*\beta_d^*\alpha_u\beta_u + \alpha_d^*\beta_u^*\alpha_d\beta_d + \alpha_d^*\beta_d^*\alpha_d\beta_u \\
= &\ \alpha_u^*\alpha_u(\beta_u^*\beta_d + \beta_d^*\beta_u) + \alpha_d^*\alpha_d(\beta_u^*\beta_d + \beta_d^*\beta_u) \\
= &\ (\beta_u^*\beta_d + \beta_d^*\beta_u)\underbrace{(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)}_{=1} \\
= &\ \beta_u^*\beta_d + \beta_d^*\beta_u
\end{aligned}$$

Remark 5. *At this stage, it's important to observe than up to a renaming ($\beta \leftarrow \alpha$), this is the same expression we had for $\langle \boldsymbol{\sigma_x} \rangle$. And it makes sense given how symmetrical the "physical" situation is. We expect to find the same thing for the two other components: if this is the case, we could then directly conclude $\langle \boldsymbol{\tau_x} \rangle^2 + \langle \boldsymbol{\tau_y} \rangle^2 + \langle \boldsymbol{\tau_z} \rangle^2 = 1$, without additional computations.*

$$
\begin{aligned}
\langle \boldsymbol{\tau_y} \rangle = \langle \Psi | \boldsymbol{\tau_y} | \Psi \rangle = &\quad \begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \\
= &\quad i \begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} -\alpha_u\beta_d \\ \alpha_u\beta_u \\ -\alpha_d\beta_d \\ \alpha_d\beta_u \end{pmatrix} \\
= &\quad i(-\alpha_u^*\beta_u^*\alpha_u\beta_d + \alpha_u^*\beta_d^*\alpha_u\beta_u - \alpha_d^*\beta_u^*\alpha_d\beta_d + \alpha_d^*\beta_d^*\alpha_d\beta_u) \\
= &\quad i(\alpha_u^*\alpha_u(-\beta_u^*\beta_d + \beta_d^*\beta_u) + \alpha_d^*\alpha_d(-\beta_u^*\beta_d + \beta_d^*\beta_u)) \\
= &\quad i(-\beta_u^*\beta_d + \beta_d^*\beta_u)\underbrace{(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)}_{=1} \\
= &\quad i(-\beta_u^*\beta_d + \beta_d^*\beta_u) \\
=_{\beta\leftarrow\alpha} &\quad \langle \boldsymbol{\sigma_y} \rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle \boldsymbol{\tau_z} \rangle = \langle \Psi | \boldsymbol{\tau_z} | \Psi \rangle = &\quad \begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \\
= &\quad \begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ -\alpha_u\beta_d \\ \alpha_d\beta_u \\ -\alpha_d\beta_d \end{pmatrix} \\
= &\quad \alpha_u^*\beta_u^*\alpha_u\beta_u - \alpha_u^*\beta_d^*\alpha_u\beta_d\alpha_d^*\beta_u^*\alpha_d\beta_u - \alpha_d^*\beta_d^*\alpha_d\beta_d \\
= &\quad \alpha_u^*\alpha_u(\beta_u^*\beta_u - \beta_d^*\beta_d) + \alpha_d^*\alpha_d(\beta_u^*\beta_u - \beta_d^*\beta_d) \\
= &\quad (\beta_u^*\beta_u - \beta_d^*\beta_d)\underbrace{(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)}_{=1} \\
= &\quad \beta_u^*\beta_u - \beta_d^*\beta_d \\
=_{\beta\leftarrow\alpha} &\quad \langle \boldsymbol{\sigma_z} \rangle
\end{aligned}
$$

Hence by our previous remark, indeed:

$$
\boxed{\langle \boldsymbol{\tau_x} \rangle^2 + \langle \boldsymbol{\tau_y} \rangle^2 + \langle \boldsymbol{\tau_z} \rangle^2 =_{\beta\leftarrow\alpha} \langle \boldsymbol{\sigma_x} \rangle^2 + \langle \boldsymbol{\sigma_y} \rangle^2 + \langle \boldsymbol{\sigma_z} \rangle^2 = 1}
$$

Moving on to the correlation:

$$
\begin{aligned}
\langle \boldsymbol{\sigma_z}\boldsymbol{\tau_z} \rangle = \langle \Psi | \boldsymbol{\sigma_z}\boldsymbol{\tau_z} | \Psi \rangle = &\quad \begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \\
= &\quad \begin{pmatrix} \alpha_u^*\beta_u^* & \alpha_u^*\beta_d^* & \alpha_d^*\beta_u^* & \alpha_d^*\beta_d^* \end{pmatrix} \begin{pmatrix} \alpha_u\beta_u \\ -\alpha_u\beta_d \\ -\alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \\
= &\quad \alpha_u^*\beta_u^*\alpha_u\beta_u - \alpha_u^*\beta_d^*\alpha_u\beta_d - \alpha_d^*\beta_u^*\alpha_d\beta_u + \alpha_d^*\beta_d^*\alpha_d\beta_d \\
= &\quad \alpha_u^*\alpha_u(\beta_u^*\beta_u - \beta_d^*\beta_d) - \alpha_d^*\alpha_d(\beta_u^*\beta_u - \beta_d^*\beta_d) \\
= &\quad \underbrace{(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)}_{=\langle \boldsymbol{\sigma_z} \rangle}\underbrace{(\beta_u^*\beta_u - \beta_d^*\beta_d)}_{=\langle \boldsymbol{\tau_z} \rangle} \\
\Leftrightarrow &\quad \boxed{\langle \boldsymbol{\sigma_z}\boldsymbol{\tau_z} \rangle - \langle \boldsymbol{\sigma_z} \rangle \langle \boldsymbol{\tau_z} \rangle = 0}
\end{aligned}
$$

Singlet state

The singlet state is characteristic of a maximally entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right)$$

This means that we *won't* be able to express this state as a tensor product of two states from S_A and S_B, as we just did for a product state.

Let's start with the wave function and normalization, for the composite space: the general form of a state vector in this space is:

$$|\Psi\rangle = \psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle$$

While the normalization condition translates to:

$$\||\Psi\rangle\| = 1 \Leftrightarrow \sqrt{\langle\Psi|\Psi\rangle} = \sqrt{\psi_{uu}^*\psi_{uu} + \psi_{ud}^*\psi_{ud} + \psi_{du}^*\psi_{du} + \psi_{dd}^*\psi_{dd}} = 1$$

But because each individual term under the square root is positive, this is equivalent to:

$$\psi_{uu}^*\psi_{uu} + \psi_{ud}^*\psi_{ud} + \psi_{du}^*\psi_{du} + \psi_{dd}^*\psi_{dd} = 1$$

For the singlet state, the wave function is:

$$\psi : \begin{cases} |uu\rangle \to & \psi_{uu} = 0 \\ |ud\rangle \to & \psi_{ud} = 1/\sqrt{2} \\ |du\rangle \to & \psi_{du} = -1/\sqrt{2} \\ |dd\rangle \to & \psi_{dd} = 0 \end{cases}$$

It's trivial to check that it's normalized.

What's the wave function for each subsystem state? Well, think about it: if there's a wave function for each subsystem, then there's a *pure*, normalized state for each subsystem, and then the composite state can be expressed as a tensor product between those two. Meaning, this composite state *would be* a product state. But, it's claimed here than the composite state is entangled, meaning, it's *not* a product state, and so we shouldn't be able to find such wave-functions for the isolated subsystems.

We've already studied in L06E03 why this particular singlet state isn't a product state, let me recall you how it went: the idea is to identify the general form of a composite state:

$$|\Psi\rangle = \psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle$$

With the general form of a product state:

$$|\Phi\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

Which yields the particular following equations systems for the singlet state:

$$\begin{cases} \psi_{ud} = \alpha_u\beta_d = & \frac{1}{\sqrt{2}} \\ \psi_{du} = \alpha_d\beta_u = & -\frac{1}{\sqrt{2}} \\ \psi_{uu} = \alpha_u\beta_u = & 0 \\ \psi_{dd} = \alpha_d\beta_d = & 0 \end{cases}$$

But consider for example the third equation, which implies that at least either $\alpha_u = 0$ or $\beta_u = 0$. In the former case, the first equation can't hold, while in the latter, the second equation can't hold. Hence the system is inconsistent, and cannot be solved.

This proves that the composite system's wave-function cannot be factorized.

So what does this mean regarding the states of the subsystems? Surely it conceptually makes sense to still talk about the existence of such states? Yes, obviously is does and that's precisely where the notion of density matrix becomes most useful[3]: to express *impure* states.

So, let's move on to density matrices then. Starting with the easiest: the composite system's: it's a pure state so we have (again, the vector/matrix representation depends implicitly on the usual ordered basis):

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let's verify the usual matrix properties for pure states:

$$\rho^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/4+1/4 & -1/4-1/4 & 0 \\ 0 & -1/4-1/4 & 1/4+1/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \rho$$

And trivially, $\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1/2 + 1/2 = 1$.

Moving on to the density matrices of the subsystems, that is, on the most accurate state description we can provide to each subsystems. In the book, we derived a formula[4]: we first introduced an arbitrary observable $\boldsymbol{L}^A$, acting on S_A, and upgraded it to the composite system: $\boldsymbol{L} = \boldsymbol{L}^A \otimes \boldsymbol{I}^B$.

Let me rework the proof, while being a bit more explicit. First, component-wise, we have:

$$\boldsymbol{L} = \boldsymbol{L}^A \otimes \boldsymbol{I}^B = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 & L_{12} & 0 \\ 0 & L_{11} & 0 & L_{12} \\ L_{21} & 0 & L_{22} & 0 \\ 0 & L_{21} & 0 & L_{22} \end{pmatrix} \Leftrightarrow L_{a'b',ab} = L^A_{a'a}\delta_{b'b}$$

Now assume we're in a composite state:

$$|\Psi\rangle = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{21} \\ \psi_{22} \end{pmatrix} ; \text{ thus: } \langle\Psi| = \begin{pmatrix} \psi_{11}^* & \psi_{12}^* & \psi_{21}^* & \psi_{22}^* \end{pmatrix}$$

We can then compute the expectation value for $\boldsymbol{L}$:

$$\begin{aligned}
\langle \boldsymbol{L} \rangle = \quad & \langle\Psi|\boldsymbol{L}|\Psi\rangle && = \sum_{a,b,a',b'} \psi^*_{a'b'} L_{a'b',ab} \psi_{ab} \\
= \quad & \sum_{a',b,a} \psi^*_{a'b'} L^A_{a',a} \psi_{ab} && = \sum_{a',a} \underbrace{\left(\sum_b \psi^*_{a'b'} \psi_{ab} \right)}_{\rho_{a'a}} L^A_{a',a} \\
= \quad & \sum_{a'} \left(\sum_a \rho_{a',a} L^A_{a',a} \right) && = \sum_{a'} \left(\sum_a \langle a|\rho^A|a'\rangle\langle a'|\boldsymbol{L}^A|a\rangle \right) \\
= \quad & \sum_{a'} \left(\sum_a \langle a'|\boldsymbol{L}^A|a\rangle\langle a|\rho^A|a'\rangle \right) && = \sum_{a'} \left(\langle a'|\boldsymbol{L}^A \underbrace{\left(\sum_a |a\rangle\langle a| \right)}_{=\boldsymbol{I}^A} \rho^A|a'\rangle \right) \\
= \quad & \sum_{a'} \left(\langle a'|\boldsymbol{L}^A\rho^A|a'\rangle \right) && =: \ \text{Tr}(\boldsymbol{L}^A\rho^A) = \text{Tr}(\rho^A\boldsymbol{L}^A)
\end{aligned}$$

Where the last equality is an usual property of the trace operator.

[3] For pure states, the density matrix ρ is isomorphic to the state $|\Psi\rangle$: $\rho = |\Psi\rangle\langle\Psi|$

[4] p204+, section 7.5 *Entanglement for two spins*

And we've already demonstrated in the book, using similar arguments, that

$$\langle \boldsymbol{L} \rangle = \langle \Psi | \boldsymbol{L} | \Psi \rangle = \mathrm{Tr}(\underbrace{|\Psi\rangle \langle \Psi|}_{=:\rho} \boldsymbol{L}) = \mathrm{Tr}(\rho \boldsymbol{L})$$

So what we've proved in the end is that:

$$\langle \boldsymbol{L} \rangle = \mathrm{Tr}(\rho \boldsymbol{L}) = \mathrm{Tr}(\rho^A \boldsymbol{L}^A) = \left\langle \boldsymbol{L}^A \right\rangle$$

That is, we've reduced the density matrix ρ on the composite system to a density matrix ρ^A on Alice's subsystem.

I'll come back to this density matrix ρ^A in a moment, but before moving on any further, let me emphasize a subtle point that I think could have been made clearer in the book. Suppose we're in a mixed state in Alice's subsystem. This means that there's some amount of chance we're in this state, or some amount we're in this other state, and so on, something like:

$$|\psi\rangle = P_1|\psi_1\rangle + P_2|\psi_2\rangle + \ldots$$

But wait a minute, each of those $|\psi_i\rangle$ is an element of the state space (the Hilbert space), so they can all be expressed as a linear combination of its basis vectors. In the context of a spin:

$$\begin{aligned} |\psi\rangle = &\quad P_1(\alpha_1|u\rangle + \beta_1|d\rangle) + P_2(\alpha_2|u\rangle + \beta_2|d\rangle + \ldots \\ = &\quad \left(\sum_i P_i\alpha_i\right)|u\rangle + \left(\sum_i P_i\beta_i\right)|d\rangle \end{aligned}$$

Assuming we renormalize that last state vector if need be, haven't we just found a wave-function describing Alice's state? But haven't we just stated that we cannot find a wave-function for Alice's state because it's a mixed state?

You could push this thinking one step further: can't we do the same thing for Bob's space, and join the two resulting states with a tensor product?

Well, there's one considerable issue with the previous reasoning, and I don't think it's clear from the book. So let me emphasize it:

$$\boxed{\text{Elements of the so called state-space } \textbf{aren't} \text{ states!}}.$$

Meaning, $|\psi\rangle$ <u>**isn't**</u> a state! What we have is the following:

$$\boxed{\text{The set of all } \textit{pure states} \text{ is } \underline{\text{isomorphic}} \text{ to } S_A = \{|\psi\rangle\}}.$$

The most careful definition of quantum system states is[5]:

$$\boxed{\text{The states of a (quantum) system are all positive, trace-class, linear maps } \rho : S_A \to S_A \text{ for which } \mathrm{Tr}\rho = 1}$$

The previous definition accounts for some refinements that will be introduced in the next chapter of Susskind's book. For example, a *trace-class* map refers to a map who has a finite trace: it's always the case in a finite dimension vector space, but divergence may occur in infinite dimension vector spaces, which are mandatory to express position observables for example.

To simplify, such maps corresponds to our density matrices, which as we've saw, can encode both pure and mixed states. We could argue on terminology regarding what a state is: a "mixed state" may only corresponds to the information we have about a state, and not to an actual, physical state, which

[5] See `https://youtu.be/GbqA9Xn_iM0?t=4453`

would justify the less strict terminology, while introducing some confusion on the use of the term ”state”.

It just so happen than for every pure state, there’s a $1 : 1$ correspondance with the elements of the Hilbert ”state” space, as $\rho = |\psi\rangle \langle\psi|$.

So we can’t just create a convex combination of ”pure states” (well, something that’s isomorphic to a pure state in our modern terminology). That’s why when density matrices were introduced in Susskind’s book, the convex combination was performed over projection operators built from pure states:

$$\rho = P_1 |\psi_1\rangle \langle\psi_1| + P_2 |\psi_2\rangle \langle\psi_2| + \dots$$

And not as I’ve just show you, directly over elements of the Hilbert space.

Now that we have a clear definition of what a state is, we can check that our matrix ρ^A really is a state: we want to prove that it’s a positive, trace-class linear map such that $\mathrm{Tr}(\rho^A) = 1$.

It’s clearly **trace-class**, because we’re in finite dimension: we have a matrix, the trace is a finite sum, it always converges.

Let me write the matrix in component form:

$$\rho^A = \begin{pmatrix} \displaystyle\sum_b \psi^*_{1b}\psi_{1b} & \displaystyle\sum_b \psi^*_{1b}\psi_{2b} \\ \displaystyle\sum_b \psi^*_{2b}\psi_{1b} & \displaystyle\sum_b \psi^*_{2b}\psi_{2b} \end{pmatrix} = \begin{pmatrix} \psi^*_{11}\psi_{11} + \psi^*_{12}\psi_{12} & \psi^*_{11}\psi_{21} + \psi^*_{12}\psi_{22} \\ \psi^*_{21}\psi_{11} + \psi^*_{22}\psi_{12} & \psi^*_{21}\psi_{21} + \psi^*_{22}\psi_{22} \end{pmatrix}$$

Let’s compute its trace:

$$\mathrm{Tr}(\rho^A) = \psi^*_{11}\psi_{11} + \psi^*_{12}\psi_{12} + \psi^*_{21}\psi_{21} + \psi^*_{22}\psi_{22} = \langle\Psi|\Psi\rangle = \boxed{1} \text{ (as } |\Psi\rangle \text{ is normalized)}$$

Lastly, as every component of the matrix is a positive real number, I guess this is enough to prove that ρ^A is **positive**.

We’re ready to move on to verify that $(\rho^A)^2 \neq \rho^A$. After a quick check, I don’t think we reach anything conclusive by carrying the computation symbolically, so let’s do it numerically:

$$(\rho^A)^2 = \begin{pmatrix} 0 \times 0 + 1/\sqrt{2} \times 1/\sqrt{2} & 0 \times (-1\sqrt{2}) + 1/\sqrt{2} \times 0 \\ (-1/\sqrt{2}) \times 0 + 0 \times (1/\sqrt{2}) & -1/\sqrt{2} \times 1/\sqrt{2} + 0 \times 0 \end{pmatrix}^2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}^2 = \frac{1}{4}\boldsymbol{I}^A \neq \rho^A$$

Clearly, $\mathrm{Tr}((\rho^A)^2) = 2(1/4) = 1/2 < 1$, as expected.

Finally, let’s crunch some numbers, using our *R* script:

$$\langle\boldsymbol{\sigma_z}\rangle = \langle\Psi|\boldsymbol{\sigma_z}|\Psi\rangle = 0; \qquad \langle\boldsymbol{\sigma_x}\rangle = \langle\Psi|\boldsymbol{\sigma_x}|\Psi\rangle = 0; \qquad \langle\boldsymbol{\sigma_y}\rangle = \langle\Psi|\boldsymbol{\sigma_y}|\Psi\rangle = 0$$

$$\langle\boldsymbol{\tau_z}\rangle = \langle\Psi|\boldsymbol{\tau_z}|\Psi\rangle = 0; \qquad \langle\boldsymbol{\tau_x}\rangle = \langle\Psi|\boldsymbol{\tau_x}|\Psi\rangle = 0; \qquad \langle\boldsymbol{\tau_y}\rangle = \langle\Psi|\boldsymbol{\tau_y}|\Psi\rangle = 0$$

$$\langle \boldsymbol{\tau_z \sigma_z} \rangle = \quad \langle \Psi | \boldsymbol{\tau_z \sigma_z} | \Psi \rangle \quad = \quad \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad = \quad -1$$

$$\langle \boldsymbol{\tau_x \sigma_x} \rangle = \quad \langle \Psi | \boldsymbol{\tau_x \sigma_x} | \Psi \rangle \quad = \quad \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad = \quad -1$$

$$\langle \boldsymbol{\tau_y \sigma_y} \rangle = \quad \langle \Psi | \boldsymbol{\tau_y \sigma_y} | \Psi \rangle \quad = \quad \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad = \quad -1$$

$$\langle \boldsymbol{\sigma_z \tau_z} \rangle = \quad \langle \Psi | \boldsymbol{\sigma_z \tau_z} | \Psi \rangle \quad = \quad \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad = \quad -1$$

The last one served to verify the following correlation:

$$\langle \boldsymbol{\sigma_z \tau_z} \rangle - \langle \boldsymbol{\sigma_z} \rangle \langle \boldsymbol{\tau_z} \rangle = -1$$

"Near-singlet" state

Starting from the following composite state:

$$|\Psi\rangle = \sqrt{0.6}|ud\rangle - \sqrt{0.4}|du\rangle$$

We can easily identify the wave-function for the composite system:

$$\psi : \begin{cases} |uu\rangle & \rightarrow \psi_{11} = 0 \\ |ud\rangle & \rightarrow \psi_{12} = \sqrt{0.6} \\ |du\rangle & \rightarrow \psi_{21} = -\sqrt{0.4} \\ |dd\rangle & \rightarrow \psi_{22} = 0 \end{cases}$$

The density matrix for the composite system naturally follows for the definition of the state:

$$\begin{aligned} \rho = &\quad |\Psi\rangle \langle \Psi| = \begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{0.6} & -\sqrt{0.4} & 0 \end{pmatrix} \\ = &\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.6 & -\sqrt{0.6 \times 0.4} & 0 \\ 0 & -\sqrt{0.6 \times 0.4} & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.6 & -\sqrt{0.24} & 0 \\ 0 & -\sqrt{0.24} & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Let's square it:

$$\rho^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.6 \times 0.6 + (-\sqrt{0.24})^2 & -0.6\sqrt{0.24} - 0.4\sqrt{0.24} & 0 \\ 0 & -0.6\sqrt{0.24} - 0.4\sqrt{0.24} & 0.4 \times 0.4 + (-\sqrt{0.24})^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.36 + 0.24 & -\sqrt{0.24} & 0 \\ 0 & -\sqrt{0.24} & 0.16 + 0.24 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \rho$$

Immediately, $\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1$.

Again, this is an entangled state: by the same (abstract) reasoning as before, we can't find a wave-function for the subsystems, and we must look for a density matrix. I'll skip the details this time:

$$\rho^A = \begin{pmatrix} \psi_{11}^* \psi_{11} + \psi_{12}^* \psi_{12} & \psi_{11}^* \psi_{21} + \psi_{12}^* \psi_{22} \\ \psi_{21}^* \psi_{11} + \psi_{22}^* \psi_{12} & \psi_{21}^* \psi_{21} + \psi_{22}^* \psi_{22} \end{pmatrix} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix}$$

Let's square it:

$$(\rho^A)^2 = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.16 \end{pmatrix} \neq \rho^A$$

Clearly, $\mathrm{Tr}((\rho^A)^2) = 0.36 + 0.16 = 0.52 < 1$.

Let's crunch some numbers again[6]; again, this has been automatically computed by the *R* script:

$$\langle \boldsymbol{\sigma_z} \rangle = \langle \Psi | \boldsymbol{\sigma_z} | \Psi \rangle = 0.2; \qquad \langle \boldsymbol{\sigma_x} \rangle = \langle \Psi | \boldsymbol{\sigma_x} | \Psi \rangle = 0; \qquad \langle \boldsymbol{\sigma_y} \rangle = \langle \Psi | \boldsymbol{\sigma_y} | \Psi \rangle = 0$$

$$\langle \boldsymbol{\tau_z} \rangle = \langle \Psi | \boldsymbol{\tau_z} | \Psi \rangle = -0.2; \qquad \langle \boldsymbol{\tau_x} \rangle = \langle \Psi | \boldsymbol{\tau_x} | \Psi \rangle = 0; \qquad \langle \boldsymbol{\tau_y} \rangle = \langle \Psi | \boldsymbol{\tau_y} | \Psi \rangle = 0$$

$$\begin{aligned}
\langle \boldsymbol{\tau_z \sigma_z} \rangle = \quad & \langle \Psi | \boldsymbol{\tau_z \sigma_z} | \Psi \rangle & = & \begin{pmatrix} 0 & \sqrt{0.6} & -\sqrt{0.4} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} & = & -1 \\
\langle \boldsymbol{\tau_x \sigma_x} \rangle = \quad & \langle \Psi | \boldsymbol{\tau_x \sigma_x} | \Psi \rangle & = & \begin{pmatrix} 0 & \sqrt{0.6} & -\sqrt{0.4} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} & \simeq & -0.9797959 \simeq -2\sqrt{0.24} \\
\langle \boldsymbol{\tau_y \sigma_y} \rangle = \quad & \langle \Psi | \boldsymbol{\tau_y \sigma_y} | \Psi \rangle & = & \begin{pmatrix} 0 & \sqrt{0.6} & -\sqrt{0.4} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} & = & -1 \\
\langle \boldsymbol{\sigma_z \tau_z} \rangle = \quad & \langle \Psi | \boldsymbol{\sigma_z \tau_z} | \Psi \rangle & = & \begin{pmatrix} 0 & \sqrt{0.6} & -\sqrt{0.4} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} & = & -1
\end{aligned}$$

The last one served to verify the following correlation:

$$\langle \boldsymbol{\sigma_z \tau_z} \rangle - \langle \boldsymbol{\sigma_z} \rangle \langle \boldsymbol{\tau_z} \rangle = -1 - 0.2 \times (-0.2) = -0.96$$

For completeness, here's the aforementioned, self-contained *R* script. You may want to look at the LaTeXsource file. I've wrote a separate article (.html) showcasing various R features used in this script.

```
#!/bin/Rscript

# Quirky, but does the job.

# Computes expectation values for a composite
# system built from two quantum spins.
#
# Used with either with 1 or 5 arguments:
#   <operator> [wave-function (uu, ud, du, dd)]
#
# Each argument is parsed and evaluated, so you can
# use "tau_x %*% sigma_x" as an operator for example.
#
# If only an operator is provided (1 arg), its 4x4 matrix form
# is displayed as LaTeX on stdout.
#
# Otherwise, evaluates the expectation value for the operator
# for a system in a state described by the wave-function.
```

[6] There are a few more than asked

```
tmp <- "/tmp/R/lib"
.libPaths(tmp)
dir.create(tmp, recursive = TRUE, mode = "0755")

loadpkg <- function(p) {
    if (!require(p, character.only = TRUE)) {
        install.packages(p, repos='http://cran.us.r-project.org', lib=tmp)
        library(p, character.only = TRUE)
    }
}

# 2x2 identity and Pauli matrices
id2 = matrix(c(1,  0,  0,  1), 2, 2)
pz  = matrix(c(1,  0,  0, -1), 2, 2)
px  = matrix(c(0,  1,  1,  0), 2, 2)
py  = matrix(c(0, 1i, -1i, 0), 2, 2)

# Ugprade subsystem spin component operators to the composite system
sigma_z = kronecker(pz, id2)
tau_z   = kronecker(id2, pz)

sigma_x = kronecker(px, id2)
tau_x   = kronecker(id2, px)

sigma_y = kronecker(py, id2)
tau_y   = kronecker(id2, py)

# Expectation value computation
avg <- function(L, psi) {
    return((Conj(t(psi)) %*% L %*% psi)[1])
}

# Evaluate CLI arguments (e.g. "interpret" "tau_z %*% sigma_z"
# as the corresponding observable)
#
# Note that we need a list (lapply), as some of the arguments will be
# vectors already.
args <- lapply(
    commandArgs(trailingOnly = TRUE),
    function(x) { return(eval(parse(text=x))) }
)

# LaTeX export
loadpkg("xtable")

# See https://tales.mbivert.com/on-exporting-r-complex-matrix-latex/
xtable <- function(x, ...) {
    if (class(x[[1]]) == "complex") {
        z <- sapply(x, function(y) {
            if (y == 0)       return(as.character(0))
            if (Im(y) == 0) return(as.character(Re(y)))

            t <- ""
            if (Re(y) != 0) t <- as.character(Re(y))

            if (Im(y) == 1) {
                if (Re(y) == 0) t <- "i"
                else            t <- paste(t, "+i")
            } else if (Im(y) == -1)
                t <- paste(t, "-i")
            else {
                if (Re(y) == 0)     t <- paste(Im(y), "i")
                else if (Im(y) > 0) t <- paste(t, "+", Im(y), "i")
                else                t <- paste(t, Im(y), "i")
            }
            return(t)
        })
        dim(z) <- dim(x)
        xtable::xtable(z, ...)
    } else
        xtable::xtable(x, ...)
}

if (length(args) == 1) {
    x <- xtable(args[[1]], align=rep("",ncol(args[[1]])+1))

    # 1.00 -> 1, in our peculiar case
    digits(x) <- xdigits(x)
    print(x,
        floating = FALSE, tabular.environment = "pmatrix",
        hline.after=NULL, include.rownames=FALSE, include.colnames=FALSE
    )
    q()
} else if (length(args) != 5) stop("Incomplete wave function")

x <- avg(args[[1]], unlist(args[2:5]))

# avoids some 0+0i; refinable
if (x == 0) {cat(0, "\n")} else {cat(x, "\n")}
```