The Theoretical Minimum

Quantum Mechanics - Solutions

L08E01

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Prove that* $\boldsymbol{X}$ *and* $\boldsymbol{D}$ *are linear operators.*

The two operators are defined on a Hilbert space $\mathcal{H}$ by:

$$\boldsymbol{X} : \begin{pmatrix} \mathcal{H} & \to & \mathcal{H} \\ \psi & \mapsto & (x \mapsto x\psi(x)) \end{pmatrix}; \qquad \boldsymbol{D} : \begin{pmatrix} \mathcal{H} & \to & \mathcal{H} \\ \psi & \mapsto & (x \mapsto \frac{d}{dx}\psi(x)) \end{pmatrix}$$

Generally speaking, an operator $\boldsymbol{L} : \mathcal{H} \to \mathcal{H}$ is said to be linear if those two axioms are verified:

$$(\forall \alpha \in \mathbb{C}),\ \boldsymbol{L}(\alpha\psi) = \alpha\boldsymbol{L}(\psi)$$

$$(\forall (\psi, \phi) \in \mathcal{H}^2),\ \boldsymbol{L}(\psi + \phi) = \boldsymbol{L}(\psi) + \boldsymbol{L}(\phi)$$

Remark 1. *It's customary in quantum-mechanics to drop the parentheses when apply an operator, i.e.* $\boldsymbol{L}\psi := \boldsymbol{L}(\psi)$*; I'll keep them here for clarity.*

Furthermore, while the authors often use "$\psi(x)$*" to denote a function, I'll be using* ψ *instead, and reserve* $\psi(x)$ *to the result of the application of* ψ *to the variable* x*, as is usual in mathematics.*

Finally, recall[1] *that addition and scalar-multiplication are defined pointwise*[2] *on functions:*

$$(\forall (\psi, \phi) \in \mathcal{H}), \quad \psi + \phi := \Big(x \mapsto (\psi + \phi)(x) := \psi(x) + \phi(x)\Big)$$

$$(\forall (\psi, \alpha) \in \mathcal{H} \times \mathbb{C}), \quad \alpha\psi := \Big(x \mapsto (\alpha\psi)(x) := \alpha\psi(x)\Big)$$

To ease notation, I'll use the same symbols for e.g. the addition of complex numbers and the (pointwise) addition of functions. Don't hesitate to label them in your mind in case of doubt.

Starting with $\boldsymbol{X}$ and the first axiom; let $x \in \mathbb{R}$, $\alpha \in \mathbb{C}$ and $\psi \in \mathcal{H}$:

$$\begin{aligned} (\boldsymbol{X}(\alpha\psi))(x) = &\quad x\alpha\psi(x) \\ = &\quad \alpha(x\psi(x)) \\ = &\quad \alpha(\boldsymbol{X}\psi)(x) \end{aligned}$$

As this is true for any x, we can conclude:

$$\boxed{\boldsymbol{X}(\alpha\psi) = \alpha\boldsymbol{X}(\psi)}$$

Moving on to the second axiom; let $x \in \mathbb{R}$ and $(\psi, \phi) \in \mathcal{H}^2$:

$$\begin{aligned} (\boldsymbol{X}(\psi + \phi))(x) = &\quad x(\psi(x) + \phi(x)) \\ = &\quad x\psi(x) + x\phi(x) \\ = &\quad (\boldsymbol{X}(\psi))(x) + (\boldsymbol{X}(\phi))(x) \\ = &\quad (\boldsymbol{X}(\psi) + \boldsymbol{X}(\phi))(x) \end{aligned}$$

[1] The authors did it a bit quickly a few pages earlier, but I've done it more carefully in L03E01

[2] https://en.wikipedia.org/wiki/Pointwise

Again, this is true for any x and thus:

$$\boxed{\boldsymbol{X}(\psi + \phi) = \boldsymbol{X}(\psi) + \boldsymbol{X}(\phi)}$$

The second operator is a little more interesting; let $x \in \mathbb{R}$, $\alpha \in \mathbb{C}$ and $\psi \in \mathcal{H}$:

$$\begin{aligned}(\boldsymbol{D}(\alpha\psi))(x) = & \ (\frac{d}{dx}\alpha\psi)(x) \\ = & \ \alpha(\frac{d}{dx}\psi)(x) \\ = & \ \alpha(\boldsymbol{D}\psi)(x)\end{aligned}$$

You may be wondering why we're allowed to shift the α outside of the differential operator. Let me clarify this a little. The (real) differentiation operator is defined as a limit:

$$\frac{d}{dx}\psi(x) := \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} =: \psi'(x)$$

So we can develop our previous equation as[3]:

$$\frac{d}{dx}(\alpha\psi(x)) = \lim_{\epsilon \to 0} \frac{\alpha\psi(x+\epsilon) - \alpha\psi(x)}{\epsilon} = \lim_{\epsilon \to 0} \alpha \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon}$$

And thus all the difficulty is in knowing wether the α can "jump" outside of the limit. And the answer is yes[4] *assuming the remaining limit exists.* Meaning we can as long as the following limit exists (it must converges to some fixed point in $\mathbb{C}$, or equivalently, is must not diverge to $\pm\infty$):

$$\lim_{\epsilon \to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon}$$

This is equivalent to saying that we can do it as long as ψ is differentiable. In a physics context, functions are often always assumed to be differentiable everywhere. Hence the first axiom indeed holds for $\boldsymbol{D}$:

$$\boxed{\boldsymbol{D}(\alpha\psi) = \alpha\boldsymbol{D}(\psi)}$$

There's an analogue reasoning for the second axiom: let $x \in \mathbb{R}$ and $(\psi, \phi) \in \mathcal{H}^2$:

$$\begin{aligned}(\boldsymbol{D}(\psi + \phi))(x) = & \ \frac{d}{dx}(\psi + \phi)(x) \\ = & \ (\frac{d}{dx}\psi + \frac{d}{dx}\phi)(x) \\ = & \ (\boldsymbol{D}(\psi) + \boldsymbol{D}(\phi))(x)\end{aligned}$$

Again we can rewrite the "questionable" line by expanding the differentiation as a limit while unwrapping the pointwise addition of functions:

$$\begin{aligned}(\frac{d}{dx}(\psi + \phi))(x) = & \ \lim_{\epsilon \to 0} \frac{(\psi + \phi)(x+\epsilon) - (\psi + \psi)(x)}{\epsilon} \\ = & \ \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon) + \phi(x+\epsilon) - (\psi(x) + \phi(x))}{\epsilon} \\ = & \ \lim_{\epsilon \to 0} \frac{\Big(\psi(x+\epsilon) - \psi(x)\Big) + \Big(\phi(x+\epsilon) - \phi(x)\Big)}{\epsilon} \\ = & \ \lim_{\epsilon \to 0} \Big(\frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} + \frac{\phi(x+\epsilon) - \phi(x)}{\epsilon}\Big)\end{aligned}$$

Again, we can split the limit of a sum to a sum of limits[5], *as long as both limits converge.* Hence the second axioms holds as long as ψ and ϕ are differentiable:

$$\boxed{\boldsymbol{D}(\psi + \phi) = \boldsymbol{D}(\psi) + \boldsymbol{D}(\phi)}$$

[3] Remember the pointwise definition of the scalar multiplication of a function.

[4] For a proof, have a look at `https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx`.

[5] There's a proof on the same website as before