The Theoretical Minimum
Quantum Mechanics - Solutions
L09E02

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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Exercise 1. *Prove Eq.* 9.10 *by expanding each site and comparing the results.*

Eq. 9.10 is:

$$[\boldsymbol{P}^2, \boldsymbol{X}] = \boldsymbol{P}[\boldsymbol{P}, \boldsymbol{X}] + [\boldsymbol{P}, \boldsymbol{X}]\boldsymbol{P}$$

We're trying to compute the "velocity" of a free particle, meaning, the time derivative of its position operator $\boldsymbol{X}$, which can be performed thanks to an earlier formula derived in section 4.9 - *Connections to classical mechanics*:

$$\frac{d}{dt}\langle\boldsymbol{X}\rangle = \frac{i}{\hbar}\langle[\boldsymbol{H}, \boldsymbol{X}]\rangle$$

To push the computation forward, we then need to compute $[\boldsymbol{H}, \boldsymbol{X}]$, which is, for a free particle, up to a constant, equivalent to $[\boldsymbol{P}^2, \boldsymbol{X}]$.

Recall that the *commutator* measures how much two operators defined on a Hilbert space commute:

$$[\boldsymbol{A}, \boldsymbol{B}] := \boldsymbol{AB} - \boldsymbol{BA}$$

Let's progressively expand both sides, keeping all expansions strictly equivalent:

$$\begin{aligned}
& [\boldsymbol{P}^2, \boldsymbol{X}] = && \boldsymbol{P}[\boldsymbol{P}, \boldsymbol{X}] + [\boldsymbol{P}, \boldsymbol{X}]\boldsymbol{P} \\
\Leftrightarrow\quad & \boldsymbol{P}^2\boldsymbol{X} - \boldsymbol{X}\boldsymbol{P}^2 = && \boldsymbol{P}(\boldsymbol{PX} - \boldsymbol{XP}) + (\boldsymbol{PX} - \boldsymbol{XP})\boldsymbol{P} \\
\Leftrightarrow\quad & \boldsymbol{PPX} - \boldsymbol{XPP} = && \boldsymbol{PPX} - \boldsymbol{PXP} + \boldsymbol{PXP} - \boldsymbol{XPP} \\
\Leftrightarrow\quad & \boldsymbol{PPX} - \boldsymbol{XPP} = && \boldsymbol{PPX} - \boldsymbol{XPP} \\
\Leftrightarrow\quad & 0 = && 0 \\
\Leftrightarrow\quad & \text{true} \quad \square
\end{aligned}$$